Backstepping Control of Linear Time Varying Systems with Known and Unknown Parameters*

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Abstract

The backstepping control design procedure has been used to develop stabilizing controllers for time invariant plants that are linear or belong to some class of nonlinear systems. The use of such a procedure to design stabilizing controllers for plants with time varying parameters has been an open problem. In this paper we consider the backstepping design procedure for linear time varying plants with known and unknown parameters. We first show that the control designs based on time invariant models are unable to guarantee stability and asymptotic tracking for linear time varying systems in general, even if the plant parameters are exactly known. Next, we involve new filter structures in the control design considering the time variations of the plant parameters. The new control design guarantees exponential convergence of the tracking error to zero if the plant parameters are exactly known. If the parameters are not precisely known but the time variations of the parameters associated with the system zeros are known, the appropriate choice of certain design parameters, without using any adaptive law, leads to closed loop stability and perfect regulation. This new control design is modified and supplemented with an update law to be applicable to linear time varying plants with unknown parameters. In the adaptive control design, the notion of structured parameter variations is used in order to include possible a priori information about the plant parameter variations. With this formulation, only the unstructured plant parameters are estimated and are required to be slowly time varying, and the structured plant parameters are allowed to have any finite speed of variation. The adaptive controller is shown to be robust with respect to the unknown but slow parameter variations in the global sense. We derive performance bounds which can be used to select certain design parameters for performance improvement.

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1 Introduction

Research on adaptive nonlinear control has been accelerated during the last decade, after introduction of a class of controllers for a set of general classes of nonlinear systems [1, 2, 3, 4, 5, 6, 7]. These controllers are based on integrator backstepping together with other nonlinear design tools such as nonlinear damping [1, 7, 8], tuning functions [7, 9], and κ and MT filters [4, 7, 10, 11]. In the absence of modelling uncertainties, these controllers achieve global boundedness, asymptotic tracking, passivity of the adaptation loop irrespective of the relative degree, and most importantly, systematic improvement of transient performance [7, 12]. These controllers have later on been modified so that they can tolerate a class of modelling uncertainties, especially high frequency unmodeled dynamics, in the global sense [13, 14, 15, 16]. The set of systems which can be controlled by these controllers includes linear time invariant (LTI) systems. Moreover, for LTI systems, these controllers bear strong parametric robustness in the sense that global stability can be achieved by choosing appropriate design parameters without the precise knowledge of the values of the plant parameters [6, 7, 17]. The corresponding adaptive controllers which deal with unknown but constant parameters [9, 7] can achieve arbitrarily improved transient performance [7, 12].

The stability properties of this class of controllers are based on the assumption that the plant parameters are time invariant (TI). In most applications, however, plant parameters may vary with time and therefore the properties of the controllers that are designed for LTI plants need to be evaluated in a time varying (TV) environment. The early attempts to design adaptive controllers for linear time varying (LTV) systems are based on the use of the certainty equivalence approach that combines a controller structure with a robust adaptive law [18, 19, 20, 21]. These controllers use the notion that slow time variations of the plant parameters act as a perturbation to the system in the same manner as unmodeled dynamics. Based on this notion, robust adaptive control schemes for LTI systems are used to guarantee signal boundedness and small tracking error of the order of the time variations of the plant parameters. Later, consideration of the TV nature of the plant and some *a priori* information about the parameter variations led to new adaptive model reference and pole placement control designs that allow the system to be fast TV [22, 23, 21]. These controllers bear the strong stability and robustness properties of their traditional counterparts for LTI systems. However, they can not guarantee good transient behavior [24, 25], and generally can not be extended to nonlinear time varying systems. In this paper, we fill this gap using the backstepping control design procedure.

We first consider LTV systems with known parameters and show that the backstepping controllers proposed in [6, 17] based on TI models are unable to guarantee stability and asymptotic tracking for LTV systems in general, even when the plant parameters are known exactly at each time t. In addition, we establish that signal boundedness can only be guaranteed if the plant parameters associated with the plant zeros vary slowly with time.

We, then, propose a new controller that guarantees stability and convergence of the tracking error to zero independent of the speed of variation of the plant parameters. The new controller uses integrator backstepping and nonlinear damping and exploits the fact that the TV plant parameters and their variations are known exactly. The stability and performance of the proposed controller is examined in the presence of parametric uncertainty. The controller guarantees signal boundedness provided that the time variations of the parametric uncertainty associated with the plant zeros are small. In particular, if we know the time variations of these parameters exactly, then exponential regulation can be achieved for zero reference input.

The new controller developed for the known parameter case based on the LTV plant model is modified and combined with an adaptive law to deal with the case of unknown plant parameters. The notion of structured parameter variations is used to incorporate any available a priori information about the modes of variation of the plant parameters into the parameter estimates. The resulting adaptive backstepping controller has the following advantages as applied to LTV plants: First, only the unstructured plant parameters are estimated and are required to be slowly TV. The structured plant parameters are allowed to have any finite speed of variation. Second, the performance bounds derived can be used to choose certain design parameters for improved performance. Furthermore, we show that the proposed adaptive controller is robust with respect to unknown but slow parameter variations. Finally, we demonstrate the properties of the developed controllers using simulations.

Notation The following notation is used throughout the paper, unless otherwise stated.

ν_i	:	the i^{th} element of vector ν
e_i	:	the i^{th} coordinate column vector in $\Re^n$
A_i^T	:	the i^{th} row of matrix A
$\|A\|$	:	the matrix Frobenius norm of A
$\hat{x}$	:	the estimate of scalar or vector signal x
$\tilde{x}$	:	the estimate error $\tilde{x} = x - \hat{x}$
$\|(f)_t\|_{p\delta}$	:	the shifted truncated $\mathcal{L}_p$-norm ($\mathcal{L}_{p\delta}$-norm), $\|(f)_t\|_{p\delta} = \left(\int_0^t e^{-\delta(t-\tau)}\|f(\tau)\|^p d\tau\right)^{(1/p)}$
$\mathcal{S}(\mu)$	:	the set $\left\{x : [0,\infty) \to \Re^n \,\middle\|\, x \in \mathcal{L}_{2e},\ \int_t^{t+T} x^T(\tau)x(\tau)d\tau \le c_0\mu T + c_1, \forall t, T \ge 0\right\}$ for a given constant $\mu \ge 0$, where $c_0, c_1 \ge 0$ are some finite constants, and c_0 is independent of μ
$\mathcal{S}(\omega)$	:	the set $\left\{x : [0,\infty) \to \Re^n \,\middle\|\, x \in \mathcal{L}_{2e},\ \int_t^{t+T} x^T(\tau)x(\tau)d\tau \le c_0 \int_t^{t+T} \omega(\tau)d\tau + c_1, \forall t, T \ge 0\right\}$ for a given $\mathcal{L}_{1e}$ function $\omega : [0,\infty) \to [0,\infty)$, where $c_0, c_1 \ge 0$ are some finite constants
$MSE(f)$	:	mean-square-error (MSE) norm of function f, $MSE(f) = \lim_{t_c\to\infty} \sup_{t\ge t_c} \frac{1}{t}\int_0^t \|f(\tau)\|^2 d\tau$
ϵ_t	:	denotes exponentially decaying to zero signals
c	:	any positive constant
I_n	:	the $n \times n$ identity matrix, sometimes use I for short

2 Problem Statement

A single-input single-output (SISO) linear plant with parameters that are smooth and bounded and have bounded derivatives which is strongly controllable and observable is topologically equivalent to the following observable canonical form [21, 26, 27]:

$$\dot{x}(t) = Ax(t) - a(t)x_1(t) + b(t)u(t) \tag{2.1}$$

$$y(t) = x_1(t) \tag{2.2}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad a(t) = \begin{bmatrix} a_{n-1}(t) \\ \vdots \\ a_0(t) \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m(t) \\ \vdots \\ b_0(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{2.3}$$

The state space model (2.1),(2.2) can also be represented in the input-output form:

$$R_l(s,t)y = Z_l(s,t)u \tag{2.4}$$

where

$$R_l(s,t) = s^n + a_{n-1}(t)s^{n-1} + \cdots + a_1(t)s + a_0(t) \tag{2.5}$$

$$Z_l(s,t) = b_m(t)s^m + b_{m-1}(t)s^{m-1} + \cdots + b_1(t)s + b_0(t) \tag{2.6}$$

are the left polynomial differential operators (PDO) [19, 21] for the plant. Equivalently, (2.4) can also be represented using the left polynomial integral operator (PIO) $R_l(t,s)^{-1}$ as

$$y = R_l(s,t)^{-1}Z_l(s,t)u \tag{2.7}$$

We make the following assumptions about the plant:

Assumption 1 The PIO $Z_l(s,t)^{-1}$ is exponentially stable with a rate at least $\gamma_p > 0$, i.e., the transition matrix $\Phi_l(t,\tau)$ corresponding to $Z_l(s,t)y = u$ satisfies $\|\Phi_l(t,\tau)\| \le ce^{-\gamma_p(t-\tau)}$ for some constant $c > 0$.

Assumption 2 The PDO's $Z_l(s,t), R_l(s,t)$ are strongly coprime $\forall t > 0$ with known orders m, n respectively and $m < n$.

Assumption 3 The plant parameters $a_i(t), b_i(t)$ are time functions which are bounded and have bounded derivatives

Assumption 4 The sign of the high frequency gain b_m is known and constant, and there exists a known constant $\underline{b} > 0$ such that $|b_m(t)| \ge \underline{b}, \forall t \ge 0$.

The control objective is to design an output feedback control law so that the closed-loop system is uniformly stable, and the plant output y tracks as close as possible a bounded, continuously differentiable reference signal y_r with measured bounded derivatives up to order $\rho = n - m$.

3 Backstepping Control: Pointwise Design

Let us assume that the plant parameters are known at each time t and use the backstepping approach to design a control law that could meet the control objective if the plant parameters were frozen in time at each point in time. We refer to this design as pointwise in time. In other words, we use the backstepping design approach developed for LTI plants to a plant that is considered for design purposes to be an LTI plant at each frozen time t in the parameter space. Then we examine whether such a design approach can lead to a controller that can handle the parameter time variations.

We consider the state dynamics equation (2.1) to construct a state estimator. Selecting a design vector $k = [\,k_1 \quad k_2 \quad \cdots \quad k_n\,]^T$ such that the matrix $A_0 \triangleq A - ke_1^T$ (or the polynomial $K(s) = s^n + k_1 s^{n-1} + \cdots + k_n$) is Hurwitz, we can rewrite (2.1) in the Laplace domain (assuming frozen parameters at each time t) as

$$x = (sI - A_0)^{-1}\,[(k-a)y + bu] + (sI - A_0)^{-1}\,[x_0] \tag{3.1}$$

Hence, we can use the following equation to derive a state estimator:

$$\hat{x} = (sI - A_0)^{-1}\,[(k-a)y + bu] + (sI - A_0)^{-1}\,[\hat{x}_0] \tag{3.2}$$

Assuming that we have no a priori information about the state vector, we can set $\hat{x}_0$ to zero and expand (3.2) as

$$\hat{x} = (sI - A_0)^{-1}[ky] - (sI - A_0)^{-1}[ay] + (sI - A_0)^{-1}[bu] \tag{3.3}$$

Treating the plant as LTI (the plant coefficients as constant), we obtain

$$\hat{x} = (sI - A_0)^{-1}[ky] - \sum_{i=0}^{n-1} a_i (sI - A_0)^{-1}[e_{n-i}y] + \sum_{i=0}^{m} b_i (sI - A_0)^{-1}[e_{n-i}u] \tag{3.4}$$

Denoting $(sI - A_0)^{-1}[ky]$, $(sI - A_0)^{-1}[e_{n-i}y]$ $(i = 0, \ldots, n-1)$, and $(sI - A_0)^{-1}[e_{n-i}u]$ $(i = 0, \cdots, m)$ by ξ_n, ξ_i, and v_i, respectively, we obtain a state estimation scheme which is very similar to [6, 7, 17]:

$$\hat{x} = \xi_n - \sum_{i=0}^{n-1} a_i \xi_i + \sum_{i=0}^{m} b_i v_i = \xi_n + \Omega\theta \tag{3.5}$$

where

$$\Omega = [\,v_m \quad \cdots \quad v_0 \quad \xi_{n-1} \quad \cdots \quad \xi_0\,] \tag{3.6}$$

$$\theta = [\,b_m \quad \cdots \quad b_0 \quad -a_{n-1} \quad \cdots \quad -a_0\,]^T \tag{3.7}$$

Noticing that $A_0 e_i = e_{i-1}$ for $i = 2, \cdots, n$ and $A_0 e_1 = -k$, we can generate ξ_i $(0 \le i \le n)$ and v_i $(0 \le i \le m)$ using the following filters:

$$\dot{\xi}_0 = A_0\xi_0 + e_n y, \quad \xi_i = A_0\xi_{i-1}\ (1 \le i \le n-1), \quad \xi_n = -A_0\xi_{n-1} \tag{3.8}$$

$$\dot{v}_0 = A_0 v_0 + e_n u, \quad v_i = A_0 v_{i-1}\ (1 \le i \le m) \tag{3.9}$$

For LTI systems, θ is a constant vector. In our case, however, θ is a vector function of time. The TV nature of θ does not affect the form of the observer equation given by (3.5) which is the same as that with θ constant. The observation error equation, however, is given by

$$\dot{\tilde{x}} = A_0\tilde{x} - \Omega\dot{\theta} \tag{3.10}$$

where $\tilde{x} = x - \hat{x}$. It is clear that in the LTI case, where θ is a constant vector, $\tilde{x}(t) \to 0$ as $t \to \infty$. Since θ is TV, i.e. $\dot{\theta} \neq 0$, $\tilde{x}(t)$ can no longer be guaranteed to go to zero in general.

Let $\bar{u} \triangleq e_\rho u$. Then, a plant parameterization to be used for control design is obtained by differentiating y and using (3.5) as follows:

$$\dot{y} = -a_{n-1}y + x_2 + b_m\bar{u}_1 = -a_{n-1}y + \hat{x}_2 + \tilde{x}_2 + b_m\bar{u}_1 = \theta^T\omega + \xi_{n,2} + \tilde{x}_2 + b_m\bar{u}_1 \tag{3.11}$$

where

$$\omega = [\, v_{m,2} \quad \cdots \quad v_{0,2} \quad \xi_{n-1,2} + y \quad \xi_{n-2,2} \quad \cdots \quad \xi_{0,2}\,]^T \tag{3.12}$$

Introducing the notation

$$h_i(s) = [\, s^i \quad \cdots \quad s \quad 1\,]^T, 1 \leq i \leq n \tag{3.13}$$

$$W_i(s) = e_i^T(sI - A_0)^{-1} \tag{3.14}$$

we can write

$$\dot{y} = \theta^T\omega + \xi_{n,2} + \eta + b_m\bar{u}_1 + \epsilon_t \tag{3.15}$$

$$\omega = \frac{s + k_1}{K(s)}\begin{bmatrix} h_m u \\ h_{n-1} y \end{bmatrix} \tag{3.16}$$

$$\eta = -W_2(s)[\Omega\dot{\theta}] \tag{3.17}$$

The time variations $\dot{\theta}$ affect the plant parametric equation (3.15) through the signal η that also depends on the filtered values of y, u. Since η is not considered to be known, it can only be treated as a modelling error. The backstepping control design based on (3.15) with $\eta = 0$ is given as follows:

$$z_1 = y - y_r \tag{3.18}$$

$$\alpha_1 = \frac{1}{b_m}(-c_1z_1 - d_1z_1 - \theta^T\bar{\omega} - \xi_{n,2} + \dot{y}_r) \tag{3.19}$$

$$z_2 = v_{m,2} - \alpha_1 \tag{3.20}$$

$$\alpha_2 = -b_m z_1 + \beta_2 - c_2z_2 - d_2(\frac{\partial\alpha_1}{\partial y})^2 z_2 \tag{3.21}$$

$$z_i = v_{m,i} - \alpha_{i-1}, \quad 3 \leq i \leq \rho \tag{3.22}$$

$$\alpha_i = -z_{i-1} + \beta_i - c_iz_i - d_i(\frac{\partial\alpha_{i-1}}{\partial y})^2 z_i, \quad 3 \leq i \leq \rho \tag{3.23}$$

$$\beta_i = \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial v_{0,j}}(-k_jv_{0,1} + v_{0,j+1}) + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)} + \frac{\partial\alpha_{i-1}}{\partial y}(\xi_{n,2} + \theta^T\omega) + \frac{\partial\alpha_{i-1}}{\partial\xi_0}(A_0\xi_0 + e_ny) + k_iv_{m,1} \tag{3.24}$$

where $c_i, \; d_i$ are positive design constants and

$$\bar{\omega} = [\,0 \quad v_{m-1,2} \quad \cdots \quad v_{0,2} \quad \xi_{n-1,2} + y \quad \xi_{n-2,2} \quad \cdots \quad \xi_{0,2}\,]^T \tag{3.25}$$

The control law is

$$u = \begin{cases} \alpha_\rho - v_{m,\rho+1} & \text{if } m > 0 \\ \alpha_\rho & \text{if } m = 0 \end{cases} \tag{3.26}$$

Let us analyze the stability properties of the controller (3.26) designed for $\eta = 0$ when applied to the TV plant with $\eta \neq 0$ by considering the following Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{\rho} z_i^2 \tag{3.27}$$

which has been used in the LTI case. Its derivative can be computed using (3.15)–(3.23) as

$$\dot{V} = -\sum_{i=1}^{\rho} c_i z_i^2 - d_1 z_1^2 - \sum_{i=2}^{\rho} d_i \frac{\partial \alpha_{i-1}}{\partial y} z_i^2 + (z_1 - \sum_{i=2}^{\rho} \frac{\partial \alpha_{i-1}}{\partial y} z_i)\eta + \epsilon_t \tag{3.28}$$

As we can see, if $\dot{\theta} = 0$, then $\eta = 0$, and V will decay to zero exponentially fast. However when $\dot{\theta} \neq 0$, due to the presence of η which depends on u and y, signal boundedness can not be guaranteed let alone asymptotic tracking unless the time variations $|\dot{\theta}|$ are sufficiently small or decay to zero with time.

The above analysis shows that the control law based on the backstepping approach for LTI plants can not guarantee stability in the presence of plant parameter variations. In the following section, we modify the backstepping control design to take into account the time varying nature of the plant parameters.

4 Backstepping Control: Time Varying Design

In this section we use the backstepping procedure for control design by taking into account the fact that the plant is TV. As before, we assume that the plant parameters are known at each time t.

4.1 Observer Design for the Time Varying Plant

The reason that the controller (3.26) can not guarantee perfect tracking or even global stability is due to the η term in the parameterization (3.15). The signal η, which acts as a perturbation to the closed loop system, is due to the time variation $\dot{\theta}$ of the parameter vector θ and depends on the closed loop signals ξ_i, v_i, and is therefore not guaranteed to be vanishing or even bounded. However, η can be constructed as follows if θ is known. Consider the filter

$$\dot{\psi} = A_0\psi - \Omega\dot{\theta} \tag{4.1}$$

and define

$$\hat{x} = \xi_n + \Omega\theta + \psi \tag{4.2}$$

It can be easily verified that $\tilde{x} = x - \hat{x}$ with $\hat{x}$ defined in (4.2) satisfies

$$\dot{\tilde{x}} = A_0\tilde{x} \tag{4.3}$$

and therefore $\hat{x}$ converges to the true state x as $t \to \infty$. If $\dot{\theta}$ is not known then ψ in (4.2) can be generated from

$$\psi = -\Omega\theta + \bar{\psi}, \quad \dot{\bar{\psi}} = A_0\bar{\psi} - ay + bu \tag{4.4}$$

which follows from (4.1) by applying the linear swapping lemma [21, 28]. Combining (4.4) with (4.2), and denoting $(sI - A_0)^{-1}[ay]$ and $(sI - A_0)^{-1}[bu]$ by ξ_a and v_a, respectively, we obtain

$$\hat{x} = \xi_n - \xi_a + v_a \tag{4.5}$$

The signals ξ_n, ξ_a, and v_a can be generated using the filters

$$\begin{aligned} \dot{\xi}_n &= A_0\xi_n + ky && (4.6) \\ \dot{\xi}_a &= A_0\xi_a + ay && (4.7) \\ \dot{v}_a &= A_0 v_a + bu && (4.8) \end{aligned}$$

The number of the filters can be reduced by defining $\bar{\xi}_a \triangleq \xi_n - \xi_a$, i.e., combining (4.6) and (4.7) as follows:

$$\dot{\bar{\xi}}_a = A_0\bar{\xi}_a + (k-a)y \tag{4.9}$$

The filter (4.8) is used for backstepping design purposes. It is easy to verify that the estimation error $\tilde{x} = x - \hat{x}$ satisfies

$$\dot{\tilde{x}} = A_0\tilde{x} \tag{4.10}$$

which indicates that $\tilde{x} \to 0$, and therefore $\hat{x} \to x$ exponentially. Using (4.8), (4.9), and (4.5), the following plant parameterization can be obtained:

$$\dot{y} = -a_{n-1}y + \bar{\xi}_{a,2} + v_{a,2} + b_m\bar{u}_1 + \epsilon_t \tag{4.11}$$

Note that the observer (4.5) incorporates the TV parameters in the filter design which gives us the desired observation error equation (4.10). In addition, only two filters are used, hence this observer scheme has the potential of reducing the mathematical complexity of the control law. However, we also note that the number of nth order filters required for observer (4.5) is three, which is one more than that in the LTI case. This is for compensating for the time variations of the plant parameters and achieve perfect tracking.

In the following subsection, we apply the backstepping procedure to design a controller for (2.1),(2.2) based on observer (4.5) and parameterization (4.11) that are more suitable for LTV plants.

4.2 Backstepping Controller Design

Let us apply the backstepping controller design steps to the LTV plant given by (4.11).

Step 1

We treat $v_{a,2}$ as the first virtual control. We define

$$z_1 = y - y_r \tag{4.12}$$

and choose

$$\alpha_1 = -c_1z_1 - d_1z_1 + a_{n-1}y - \bar{\xi}_{a,2} + \dot{y}_r \tag{4.13}$$

Step $2 \le i \le \rho$

In each subsequent step, we individually treat $v_{a,i}$ as the virtual control and therefore the associated error signals and stabilizing functions are recursively defined as

$$z_i = v_{a,i} - \alpha_{i-1} \tag{4.14}$$

$$\alpha_i = -z_{i-1} - c_i z_i - d_i(\frac{\partial \alpha_{i-1}}{\partial y})^2 z_i + \frac{\partial \alpha_{i-1}}{\partial y}(-a_{n-1}y + \bar{\xi}_{a,2} + v_{a,2}) + k_i v_{a,1}$$
$$+\frac{\partial \alpha_{i-1}}{\partial \bar{\xi}_a}\left(A_0\bar{\xi}_a + (k-a)y\right) + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial v_{a,j}}(-k_j v_{a,1} + v_{a,j+1})$$
$$+\sum_{j=0}^{i-1}\frac{\partial \alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)} + \sum_{j=1}^{i-1}\sum_{l=0}^{i-j-1}\frac{\partial \alpha_{i-1}}{\partial a_{n-j}^{(l)}}a_{n-j}^{(l+1)} \tag{4.15}$$

In the final step, when we differentiate $v_{a,\rho}$, the control u appears in the form of $b_m u$. Therefore we can design the control law as

$$u = \begin{cases} \frac{\alpha_\rho - v_{m,\rho+1}}{b_m} & \text{if } m > 0 \\ \frac{\alpha_\rho}{b_m} & \text{if } m = 0 \end{cases} \tag{4.16}$$

where α_ρ is the ρth stabilizing function bearing the same definition as (4.15), which completes the design. The stability of the control law (4.16) can be established by using the Lyapunov function

$$V = \sum_{i=1}^{\rho} z_i^2 \tag{4.17}$$

whose derivative is given by

$$\dot{V} = -\sum_{i=1}^{\rho} c_i z_i^2 + -d_1 z_1^2 - \sum_{i=2}^{\rho} d_i(\frac{\partial \alpha_{i-1}}{\partial y})^2 z_i^2 + \epsilon_t \leq -\sum_{i=1}^{\rho} c_i z_i^2 + \epsilon_t \tag{4.18}$$

From (4.17), (4.18), it follows that $z_i \to 0$ as $t \to \infty$ exponentially fast. Hence the tracking error converges to zero exponentially, and the signals y and $\bar{\xi}_a$ are uniformly bounded. To establish the boundedness of u, we first see that $v_{a,1}$ is bounded due to the exponential stability of $Z_l(s,t)^{-1}$. Using the boundedness of z_i, we can recursively establish that α_1, $v_{a,2}$, α_2, $v_{a,3}$, $\cdots$, α_ρ, $v_{m,\rho+1}$ and finally u are all bounded. We summarize our results using the following theorem:

Theorem 4.1 *For the LTV plant (2.1),(2.2) with Assumptions 1-4, the controller given by (4.16) guarantees that the closed-loop system is internally stable, and the tracking error converges to zero exponentially fast.*

We note that the traditional polynomial based model reference controller scheme cannot guarantee perfect tracking when the TV plant parameters are completely known [19], and that a different filter structure has been proposed in [22] to resolve this problem for the model reference control case. In our case perfect tracking is achieved using two nth order filters. We also notice that by using only the signals $\bar{\xi}_a$ and v_a instead of a series of ξ_i's and v_i's, we have significantly reduced the mathematical complexity of the stabilizing functions. This is another advantage of the new controller.

The results of Theorem 4.1 are based on the assumption that the TV plant parameters are known for all $t \geq 0$. In many applications, this assumption may not hold. Consequently, it is of interest to examine the robustness of the proposed controller when only some nominal (TV) values of the plant parameters are known. In the following section, we address the robustness of the controller with respect to parametric uncertainties.

5 Parametric Robustness of the Proposed Controller

In the previous sections, we assumed that the plant parameters were known exactly. A natural question one may ask is: What if the plant parameters are not precisely known? That is, what if the actual plant dynamics are described by

$$\dot{x}(t) \quad = \quad Ax(t) - a^*(t)x_1(t) + b^*(t)u(t) \tag{5.1}$$

instead of (2.1), and there exist errors $\tilde{a}(t) = a(t) - a^*(t)$, $\tilde{b}(t) = b(t) - b^*(t)$ between the actual parameters $a^*(t) = [a^*_{n-1}(t), \cdots, a^*_0(t)]^T, b^*(t) = [b^*_m(t), \cdots, b^*_0(t)]^T$ and the parameters $a(t), b(t)$ used in the control design? This section answers this question.

Due to the parameter errors $\tilde{a}$ and $\tilde{b}$, the observer described in (4.2) or (4.5) is no longer a true state observer. In fact, if we substitute $a = \tilde{a} + a^*$ and $b = \tilde{b} + b^*$ in (4.5), we obtain

$$\dot{\tilde{x}} = A_0\tilde{x} + \tilde{a}y - \tilde{b}u \tag{5.2}$$

Applying the swapping lemma, we have

$$\tilde{x} = -\Omega\tilde{\theta} + (sI - A_0)^{-1}[\Omega\dot{\tilde{\theta}}]$$

where $\tilde{\theta}(t) \triangleq \theta(t) - \theta^*(t) = [b_m(t), \cdots, b_0(t), a_{n-1}(t), \cdots, a_0(t)]^T - [b^*_m(t), \cdots, b^*_0(t), a^*_{n-1}(t), \cdots, a^*_0(t)]^T$. The corresponding plant parameterization is

$$\begin{array}{rcl} \dot{y} & = & -a_{n-1}y + \bar{\xi}_{a,2} + v_{a,2} - \tilde{\theta}^T\omega + \eta_y + \eta_u + \epsilon_t \\ \eta_y & = & -W_2(s)[\sum_{i=0}^{n-1}\dot{\tilde{a}}_i\xi_i], \quad \eta_u = W_2(s)[\sum_{i=0}^{m}\dot{\tilde{b}}_i v_i] \end{array} \tag{5.3}$$

A lengthy analysis based on the plant parameterization (5.3) results in the following theorem, which establishes the robustness properties of the controller (4.16) with respect to the parametric uncertainties.

Theorem 5.1 *Assume that the parameter error $\tilde{b}_m(t) = b_m(t) - b^*_m(t)$ remains small for all time t in the sense that $\exists\beta_1 \geq \beta_0 \geq 0$ such that $\beta_0 \leq 1 - \tilde{b}_m(t) \leq \beta_1, \forall t \geq 0$. Furthermore, select the design parameters c_i, d_i to satisfy*

$$c_2 \; > \; \bar{\beta}_1^2, \tag{5.4}$$

$$c_1 \; > \; \beta_1\left(\frac{\bar{\beta}_1^2}{2c_2} + \left(\frac{1}{d_1\beta_0} + \bar{\beta}_2^2\right)\left(\|c_q\|^2_\infty\|P_q b_q\|^2_\infty + \|P_0\sum_{i=0}^{n-1}\dot{\tilde{a}}_i A_0^i\|^2_\infty\right) + \frac{\|d_q\|^2_\infty}{d_1\beta_0} + \bar{\beta}_2^2\right), \tag{5.5}$$

$$d_i \; \geq \; (\rho - 1)\max\left(2\|d_q\|^2_\infty + (c_1 + d_1)^2, 2\right) \quad \text{for } i = 2, \cdots, \rho, \tag{5.6}$$

where $\{A_q(t), b_q(t), c_q(t), d_q(t)\}$ is a state space realization of $Q(s,t) \triangleq -\sum_{i=0}^{n-1}\tilde{a}_i s^i(s+k_1)K(s)^{-1} + \sum_{i=0}^{m-1}\tilde{b}_i s^i(s+k_1)(Z_l(s,t)K(s))^{-1}R_l(s,t) + \tilde{b}_m s\left((s^{m-1}(s+k_1))(Z_l(s,t)K(s))^{-1}R_l(s,t) - 1\right)$, $P_q(t) \in \mathcal{L}_\infty$ is the positive definite symmetric solution to the Lyapunov equation $\dot{P}_q(t) + A_q(t)^T P_q(t) + P_q(t)A_q(t) = -2I$, and $P_0 = P_0^T > 0$ is a constant positive definite matrix satisfying $A_0^T P_0 + P_0 A_0 = -2I$. Then, there exists a $\mu^* > 0$ such that $\forall\mu \in [0, \mu^*)$, if the derivatives of the parameter errors $\tilde{b}_i$ satisfy

$$\int_t^{t+T}\sum_{i=0}^{m}|\dot{\tilde{b}}_i(\tau)|^{p_i}d\tau < c_\mu + \mu T \quad \forall t, T > 0 \tag{5.7}$$

for some $p_i \geq 1$, $c_\mu > 0$, then the closed loop system (5.1), (2.2),(4.16) is uniformly stable, and the tracking error is of the order of μ in the mean square sense.

Proof: The proof is long and technical, and is presented in Appendix A. ■

Remark 5.1 *Theorem 5.1 indicates that the uncertainty in the a_i parameters can be counteracted by increasing the values of the design parameters, c_1, c_2, $d_2, \cdots, d_\rho$, in particular. Note that for μ sufficiently small, we can find design parameters c_i, d_i such that (5.4), (5.5), and (5.6) hold. As for the b_i parameters, only the derivatives of the parameter errors have to be small, not necessarily the parameter errors themselves. This suggests that the backstepping controller has strong parametric robustness as opposed to the traditional ones. A special situation is the LTI case, where the time variations of the plant parameters or parameter errors are zero. Then we reach the same conclusion as in [6]. In this case, if the reference input is zero, then exponential regulation is achieved.*

In this section, we assume that the nominal values of the TV plant parameters are known. For stability, we require that the parametric uncertainty is small in the sense that the time variation (first time derivative) of the parametric uncertainty is small in the average sense, i.e., small most of the time. In the following section, we combine the proposed controller designed for LTV plants with known parameters with an appropriate parameter estimation scheme to deal with the case of unknown plant parameters.

6 Adaptive Backstepping Control

In the previous sections, it is assumed that the plant parameters are known precisely or with some small error. In this section, we consider these parameters as unknown functions of time which satisfy Assumptions 1-4. In order to incorporate any available a priori information about the modes of variation of the plant parameters, we use the structured parameter variations representation[21], i.e., we assume that the plant parameters $a_i(t)$, $b_i(t)$ have the following known structure:

$$\begin{bmatrix} b(t) \\ -a(t) \end{bmatrix} = \Psi(t)\theta_u(t) + \theta_0(t) = \begin{bmatrix} \Psi_b(t) \\ \Psi_a(t) \end{bmatrix} \theta_u(t) + \begin{bmatrix} \theta_{0b}(t) \\ \theta_{0a}(t) \end{bmatrix} \tag{6.1}$$

where $\Psi_a \in \Re^{n\times N}$, $\Psi_b \in \Re^{n\times N}$ form the decomposition of $\Psi \in \Re^{2n\times N}$ which is a matrix of known time functions, $\theta_u(t) \in \Re^N$ is the unstructured parameter vector that is unknown; $\theta_0(t) \in \Re^{2n}$ is a known parameter vector which can be decomposed to $\theta_{0a}(t), \theta_{0b}(t) \in \Re^n$. Note that the leading $n-m-1$ rows of Ψ_b, θ_{0b} are zeros. Furthermore, we assume the following about the leading nonzero term of θ_{0b} and the unstructured parameters.

Assumption 5 The sign of $\theta_{0b\rho}$ is the same as the sign of b_m for all $t \geq 0$. Moreover, the unstructured parameter vector θ_u is differentiable with respect to time and satisfies $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{S}(\mu) \cap \mathcal{L}_\infty$, i.e., the signals $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u$ are bounded and

$$\int_t^{t+T} |\dot{\theta}_u(\tau)|d\tau \leq c + \mu T, \quad \int_t^{t+T} |\dot{\theta}_u(\tau)|^2 d\tau \leq c + \mu T, \quad \forall t, T \geq 0 \tag{6.2}$$

for some $c \geq 0$ and $\mu > 0$, a "small" scalar.

Assumption 5 requires the mean square value of the time variations to be of order μ, where μ will be required to be small. Next, we exploit the TV model based filter design of Section 4 to construct a state estimator for the unknown parameter case. Using (6.1), we can rewrite (4.8) and (4.9) as

$$
\begin{aligned}
\bar{\xi}_a &= (sI - A_0)^{-1}\left[(k + \psi_a\theta_u + \theta_{0a})y\right] = (sI - A_0)^{-1}\left[(k+\theta_{0a})y\right] + (sI - A_0)^{-1}\left[(\psi_a y)\theta_u\right] \\
v_a &= (sI - A_0)^{-1}\left[(\psi_b\theta_u + \theta_{0b})u\right] = (sI - A_0)^{-1}\left[\theta_{0b}u\right] + (sI - A_0)^{-1}\left[(\psi_b u)\theta_u\right]
\end{aligned}
$$

Applying the linear swapping lemma, we get

$$
\begin{aligned}
\bar{\xi}_a &= \zeta + \Xi\theta_u - (sI - A_0)^{-1}\left[\Xi\dot{\theta}_u\right] \\
v_a &= \varsigma + \Upsilon\theta_u - (sI - A_0)^{-1}\left[\Upsilon\dot{\theta}_u\right]
\end{aligned}
$$

where $\zeta \triangleq (sI - A)^{-1}[(k + \theta_{0a})y]$, $\varsigma \triangleq (sI - A)^{-1}[\theta_{0b}u]$, $\Xi \triangleq (sI - A)^{-1}[\psi_a y]$, and $\Upsilon \triangleq (sI - A)^{-1}[\psi_b u]$. Hence, constructing the filters

$$\dot{\Xi} = A_0\Xi + \Psi_a y \tag{6.3}$$

$$\dot{\Upsilon} = A_0\Upsilon + \Psi_b u \tag{6.4}$$

$$\dot{\zeta} = A_0\zeta + (k + \theta_{0a})y \tag{6.5}$$

$$\dot{\varsigma} = A_0\varsigma + \theta_{0b}u \tag{6.6}$$

where $\Xi, \Upsilon \in \Re^{n\times N}$, $\zeta, \varsigma \in \Re^n$ $k = [\,k_1 \quad k_2 \quad \cdots \quad k_n\,]^T$ is such that $A_0 = A - ke_1^T$ is exponentially stable with stability margin γ_k, i.e., $A_0 - \gamma_k I$ has all eigenvalues non-positive, and considering the "virtual observer"

$$\hat{x} = \zeta + \varsigma + \Xi\theta_u + \Upsilon\theta_u \tag{6.7}$$

it is straightforward to verify that the observation error $\tilde{x} = x - \hat{x}$ satisfies

$$\dot{\tilde{x}} = A_0\tilde{x} - (\Xi + \Upsilon)\dot{\theta}_u \tag{6.8}$$

When $\dot{\theta}_u = 0$, $\tilde{x}(t) \to 0$ as $t \to \infty$. Hence (6.7) is a true state observer for (2.1),(2.2) when the parameter vector θ_u is known and constant. If θ_u is not constant, then the observation error $\tilde{x}$ is non-vanishing and is represented in the following transfer function form

$$\tilde{x} = -(sI - A_0)^{-1}[(\Xi + \Upsilon)\dot{\theta}_u] + \epsilon_t \tag{6.9}$$

Using (6.9), we can obtain the following plant parameterization

$$
\begin{aligned}
\dot{y} &= -a_{n-1}y + x_2 + b_m\bar{u}_1 = \Psi_{a1}^T\theta_u y + \theta_{0a1}y + \hat{x}_2 + \tilde{x}_2 + b_m\bar{u}_1 \\
&= w_0 + \omega^T\theta_u + \tilde{x}_2 + b_m\bar{u}_1
\end{aligned} \tag{6.10}
$$

where

$$\omega = \Psi_{a1}y + \Xi_2 + \Upsilon_2, \quad w_0 = \theta_{0a1}y + \zeta_2 + \varsigma_2 \tag{6.11}$$

$$\tilde{x}_2 = -W_2(s)[(\Xi + \Upsilon)\dot{\theta}_u] + \epsilon_t \tag{6.12}$$

$$W_2(s) = e_2^T(sI - A_0)^{-1} \tag{6.13}$$

$$\bar{u} = e_\rho u \tag{6.14}$$

Similar to the pointwise design of Section 3, the parameterization (6.10) appears to be in the same form as the LTI case [7, 17] except for the $\dot{\theta}_u$ term in $\tilde{x}_2$, and is suitable for applying the adaptive backstepping design.

Remark 6.1 *Note that (6.1) covers the general case including the fully structured, unstructured and known parameter cases. If the parameters are unstructured, then we simply have* $\Psi(t) = I_{2n\times 2n}$*. If the parameters are fully structured, then* θ_u *is constant but unknown. The case where* $\theta_u \equiv 0$ *corresponds to the known parameter case.*

Remark 6.2 *Even though the filters (6.3), (6.4) appear to be of high order since both* Ψ_a, Ψ_b *are* $n \times N$ *matrices, the actual implementation of these two filters can be of lower order, depending on the elements of* Ψ_a, Ψ_b*. In general, if* Ψ_a *contains* N_a *linearly independent time functions, then the* Ξ *filter can be obtained using* N_a *nth order filters. For example, suppose*

$$\Psi_a(t) = \sum_{i=1}^{N_a} Q[i]\varphi_i(t), \quad \Psi_b(t) = \sum_{i=1}^{N_b} \bar{Q}[i]\bar{\varphi}_i(t)$$

where $Q[i], \bar{Q}[i]$*'s are* $n \times N$ *constant matrices,* $\{\varphi_i(t)\}, \{\bar{\varphi}_i(t)\}$ *are linearly independent scalar time functions, then it suffices to implement*

$$\begin{aligned} \dot{\psi}_i &= A_0\psi_i + e_n\varphi_i y, \quad i = 1, 2, \cdots, N_a \\ \dot{\bar{\psi}}_i &= A_0\bar{\psi}_i + e_n\bar{\varphi}_i u, \quad i = 1, 2, \cdots, N_b \end{aligned}$$

where $\psi_i, \bar{\psi}_i \in \Re^n$*. The matrix* Ξ, Υ *can be realized using* $\{\psi_i\}, \{\bar{\psi}_i\}$ *as*

$$\begin{aligned} \Xi &= \sum_{i=1}^{N_a} [\ A_0^{n-1}\psi_i \ \ \cdots \ \ A_0\psi_i \ \ \psi_i\ ]Q[i] \\ \Upsilon &= \sum_{i=1}^{N_b} [\ A_0^{n-1}\bar{\psi}_i \ \ \cdots \ \ A_0\bar{\psi}_i \ \ \bar{\psi}_i\ ]\bar{Q}[i] \end{aligned}$$

Moreover, if $k + \theta_{0a}$, θ_{0b} *can be linearly represented by* $\{\varphi_i\}$, $\{\bar{\varphi}_i\}$*, respectively, say*

$$k + \theta_{0a} = \sum_{i=1}^{N_a} \Theta_{ai}\varphi_i(t), \quad \theta_{0b} = \sum_{i=1}^{N_b} \Theta_{bi}\bar{\varphi}_i(t)$$

where Θ_{ai}, Θ_{bi} *are constant vectors of length* n*, then the filter signals* ζ *and* ς *can be obtained through* $\{\psi_i\}$, $\{\bar{\psi}_i\}$ *as follows*

$$\begin{aligned} \zeta &= \sum_{i=1}^{N_a} [\ A_0^{n-1}\psi_i \ \ \cdots \ \ A_0\psi_i \ \ \psi_i\ ]\Theta_{ai} \\ \varsigma &= \sum_{i=1}^{N_b} [\ A_0^{n-1}\bar{\psi}_i \ \ \cdots \ \ A_0\bar{\psi}_i \ \ \bar{\psi}_i\ ]\Theta_{bi} \end{aligned}$$

In this case the filters are of total order $2n(N_a + N_b)$*. In particular, when the parameters are not structured,* $N_a = N_b = 1$ *and* $\theta_{0a} = \theta_{0b} = 0$*, then the total filter order is* $2n$*, which is similar to the LTI case [6, 7, 17].*

6.1 Certainty Equivalence Control Law

The controller design follows the same procedure as in the known parameter case. The idea is to recursively treat $\Upsilon_i^T\theta_u + \varsigma_i$ as a virtual control signal, and apply the backstepping procedure using the certainty equivalence, i.e., replacing the unknown parameter vector θ_u with its on-line estimate $\hat{\theta}_u$. The design steps are as follows:

Step 1

$$
\begin{aligned}
z_1 &= y - y_r && (6.15)\\
\alpha_1 &= -c_1 z_1 - d_1 z_1 - \theta_{0a1} y - \zeta_2 - (\Psi_{a1}^T y + \Xi_2^T)\hat{\theta}_u + \dot{y}_r && (6.16)
\end{aligned}
$$

Step i: $2 \le i \le \rho$

$$
\begin{aligned}
z_i &= \Upsilon_i^T\hat{\theta}_u + \varsigma_i - \alpha_{i-1} && (6.17)\\
\alpha_i &= -z_{i-1} - c_i z_i - d_i s_i^2 z_i + k_i(\Upsilon_1^T\hat{\theta}_u + \varsigma_1) + \frac{\partial\alpha_{i-1}}{\partial y}(w_0 + \omega^T\hat{\theta}_u) + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\Upsilon_j}(-k_j\Upsilon_1 + \Upsilon_{j+1})\\
&\quad + \sum_{j=1}^{i}\frac{\partial\alpha_{i-1}}{\partial\zeta_j}(-k_j\zeta_1 + \zeta_{j+1} + (k+\theta_{0a})_j y) + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\varsigma_j}(-k_j\varsigma_1 + \varsigma_{j+1})\\
&\quad + \sum_{j=1}^{i}\frac{\partial\alpha_{i-1}}{\partial\Xi_j}(-k_j\Xi_1 + \Xi_{j+1} + \Psi_{aj}y) + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)} && (6.18)\\
s_i^2 &= (\frac{\partial\alpha_{i-1}}{\partial y})^2 + \left\|\frac{\partial^2\alpha_{i-1}}{\partial\Xi\partial\hat{\theta}_u}\right\|^2 + \left\|\frac{\partial^2\alpha_{i-1}}{\partial\Upsilon\partial\hat{\theta}_u}\right\|^2 + \left\|\frac{\partial^2\alpha_{i-1}}{\partial\zeta\partial\hat{\theta}_u}\right\|^2 + \left\|\frac{\partial^2\alpha_{i-1}}{\partial\varsigma\partial\hat{\theta}_u}\right\|^2 + \left\|\frac{\partial^2\alpha_{i-1}}{\partial y\partial\hat{\theta}_u}\right\|^2 + 1 && (6.19)
\end{aligned}
$$

In Step ρ, the control u appears in the form of $\hat{b}_m u \triangleq (\Psi_{b\rho}^T\hat{\theta}_u + \theta_{0b\rho})u$, therefore the control law can be chosen as

$$
u = \delta_u \frac{\frac{\partial\alpha_{\rho-1}}{\partial\hat{\theta}_u} - \Upsilon_\rho^T}{\hat{b}_m}\dot{\hat{\theta}}_u + \begin{cases} \frac{\alpha_\rho}{\hat{b}_m} & \text{if } m = 0 \\ \frac{\alpha_\rho - \Upsilon_{\rho+1}^T\hat{\theta}_u - \varsigma_{\rho+1}}{\hat{b}_m} & \text{if } m \ge 1 \end{cases} \tag{6.20}
$$

where δ_u is either 0 or 1, the latter corresponding to the case where $\dot{\hat{\theta}}_u$ appears explicitly in the control law. Note that for the control law (6.20) to exist, the adaptive law must assure that $|\Psi_{b\rho}^T\hat{\theta}_u + \theta_{0b\rho}| = |\hat{b}_m(t)| \ge \underline{b}, \forall t \ge 0$.

With the control law (6.20), the corresponding error system is given by

$$
\dot{z} = A_z(z, \hat{\theta}_u, t)z + b_z(z, \hat{\theta}_u, t)(\tilde{\theta}_u^T\omega + \tilde{x}_2) + D_z(z, \hat{\theta}_u, t)\dot{\hat{\theta}}_u \tag{6.21}
$$

where $\tilde{\theta}_u \triangleq \theta_u - \hat{\theta}_u$

$$
\begin{aligned}
z &= \begin{bmatrix} z_1 & z_2 & \cdots & z_\rho \end{bmatrix}^T && (6.22)\\
A_z &= \begin{bmatrix} -c_1 - d_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - d_2 s_2^2 & 1 & \ddots & \vdots \\ 0 & -1 & -c_3 - d_3 s_3^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -c_\rho - d_\rho s_\rho^2 \end{bmatrix} && (6.23)
\end{aligned}
$$

$$b_z = [1 \quad \kappa_2 \quad \kappa_3 \quad \cdots \quad \kappa_\rho]^T \tag{6.24}$$
$$D_z = [0 \quad \lambda_2^T \quad \lambda_3^T \quad \cdots \quad \lambda_{\rho-1}^T \quad \lambda_\rho^T(1-\delta_u)]^T \tag{6.25}$$
$$\kappa_i = -\frac{\partial \alpha_{i-1}}{\partial y}, \quad \lambda_i = -\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_u} + \Upsilon_i^T \tag{6.26}$$

6.2 Adaptive Law with an Auxiliary Filter

The adaptive law for generating the parameter estimates used in the control law (6.20) is based on the idea of introducing an auxiliary filter to counteract the effect of $\dot{\hat{\theta}}_u$ term in the z equation, therefore ending up with a new error system that is suitable for synthesizing an adaptive law based on a Lyapunov function. We define the following auxiliary filter:

$$\dot{\phi} = A_z\phi - D_z\dot{\hat{\theta}}_u \tag{6.27}$$

and auxiliary error signal

$$\epsilon_1 = z + \phi \tag{6.28}$$

Then the error signal ϵ_1 satisfies the following equation

$$\dot{\epsilon}_1 = A_z\epsilon_1 + b_z(\tilde{\theta}_u^T\omega + \tilde{x}_2) \tag{6.29}$$

Since $\tilde{x}_2$ is not guaranteed to be bounded, we introduce the following normalizing signal

$$m_s = \sqrt{1 + \delta_1(\|u_t\|_{2\delta_0}^2 + \|y_t\|_{2\delta_0}^2)} \tag{6.30}$$

where $0 < \delta_0 < 2\min\{\gamma_k, c_i\}$ and $\delta_1 > 0$ are design constants. Some important properties of m_s are given by the following lemma:

Lemma 6.1 *We have*

$$\frac{y}{m_s}, \frac{\Xi}{m_s}, \frac{\Upsilon}{m_s}, \frac{\zeta}{m_s}, \frac{\varsigma}{m_s} \in \mathcal{L}_\infty \tag{6.31}$$
$$\frac{|\tilde{x}_2|}{m_s} \le c\|\dot{\theta}_u\|_{2\delta^*} \quad \forall \delta^* \in (0, \delta_0) \tag{6.32}$$
$$\frac{\dot{m}_s}{m_s} \ge -\frac{\delta_0}{2} \tag{6.33}$$

where c is a positive constant.

Proof: The state equation (2.1) can be rewritten as

$$\dot{x}(t) = (A - ke_1^T)x(t) + (k - a(t))y(t) + b(t)u(t) \tag{6.34}$$

from which we obtain

$$y(t) = e_1^T(sI - A - ke_1^T)\,[b(t) \quad k - a(t)]\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \tag{6.35}$$

Observing (6.3)-(6.6),(6.35), we see that y,Ξ,Υ, ζ, and ς can be represented as outputs of stable filters with inputs y and u. Hence the result (6.31) follows immediately. Similarly, (6.9) implies that $\frac{|\tilde{x}_2|}{\|(\Xi+\Upsilon)\dot{\theta}_u\|_{2\delta^*}} \in \mathcal{L}_\infty$, which together with (6.31) leads to (6.32). Finally, using the inequality $m_s(t+\tau) \ge e^{-\frac{\delta_0}{2}\tau}m_s(t), \forall t, \tau \ge 0$, we obtain $\frac{\dot{m}_s}{m_s} = \lim_{\tau\to 0}\frac{m_s(t+\tau)-m_s(t)}{\tau m_s(t)} \ge \lim_{\tau\to 0}\frac{e^{-\frac{\delta_0}{2}\tau}-1}{\tau} = -\frac{\delta_0}{2}$. ■

Now define the normalized estimation error

$$\epsilon = \frac{\epsilon_1}{m_s} \tag{6.36}$$

Then ϵ satisfies

$$\dot{\epsilon} = \left(A_z - \frac{\dot{m}_s}{m_s}\right)\epsilon + b_z\left(\tilde{\theta}_u^T\frac{\omega}{m_s} + \frac{\tilde{x}_2}{m_s}\right) \tag{6.37}$$

By considering (6.37), (6.33) and the following Lyapunov-like function

$$V_\theta = \frac{1}{2}\epsilon^T\epsilon + \frac{1}{2}\tilde{\theta}_u^T\Gamma^{-1}\tilde{\theta}_u, \tag{6.38}$$

the following robust adaptive law can be chosen:

$$\dot{\hat{\theta}}_u = Pr[\Gamma b_z^T\epsilon\frac{\omega}{m_s} - \Gamma\sigma(\hat{\theta}_u - \theta_c)] \tag{6.39}$$

where θ_c is the nominal value of θ_u, i.e., $\theta_u(t)$ is expected to be close to the constant vector θ_c, Γ is a positive definite symmetric gain matrix, Pr is the projection operator to project $\hat{\theta}_u$ along the boundary $|\hat{b}_m| = \text{sgn}(b_m)\Psi_{b\rho}^T\hat{\theta}_u + |\theta_{0b\rho}| \geq \underline{b}$, which is defined as

$$Pr[\nu] = \begin{cases} (I - \Gamma\frac{\Psi_{b\rho}\Psi_{b\rho}^T}{\Psi_{b\rho}^T\Gamma\Psi_{b\rho}})\nu & \text{if } |\hat{b}_m| = \underline{b} \text{ and } \text{sgn}(b_m)\Psi_{b\rho}^T\nu \leq 0 \\ \nu & \text{otherwise} \end{cases}$$

and σ is the leakage coefficient [28] defined as

$$\sigma = \begin{cases} \sigma_0 & \text{if } |\hat{\theta}_u - \theta_c| > 2M \\ \sigma_0(\frac{|\hat{\theta}_u - \theta_c|}{M} - 1) & \text{if } M < |\hat{\theta}_u - \theta_c| \leq 2M \\ 0 & \text{if } |\hat{\theta}_u - \theta_c| \leq M \end{cases} \tag{6.40}$$

M is a known upper bound for $|\theta_u - \theta_c|$, and $\sigma_0 > 0$ is a small constant.

The stability properties of the adaptive law are described by the following lemma:

Lemma 6.2 *Assume that* $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{S}(\mu) \cap \mathcal{L}_\infty$, *then the adaptive law (6.39) guarantees that*

(i) $|\hat{b}_m| \geq \underline{b}$, $\sigma\tilde{\theta}_u^T(\hat{\theta}_u - \theta_c) \leq 0$, $\sigma|\hat{\theta}_u - \theta_c| \leq \frac{-\sigma\tilde{\theta}_u^T(\hat{\theta}_u - \theta_c)}{M - |\theta_u - \theta_c|}$, $\forall t \geq 0$.

(ii) $\hat{\theta}_u, \tilde{\theta}_u, \epsilon, \dot{\hat{\theta}}_u, \frac{\phi}{m_s} \in \mathcal{L}_\infty$

(iii) $\epsilon, b_z\epsilon, \sqrt{-\sigma\tilde{\theta}_u^T\hat{\theta}_u}, \sigma\hat{\theta}_u, \dot{\hat{\theta}}_u, \frac{\phi}{m_s} \in \mathcal{S}(\mu)$. *In particular, if* $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_\infty$, *then* $\epsilon, \frac{\phi}{m_s} \in \mathcal{L}_2$ *and* $\epsilon, \frac{\phi}{m_s} \to 0$ *as* $t \to \infty$.

Proof: The properties in (i) are direct consequences of the projection and switching σ-modification, see [28].

To prove (ii) and (iii), let us consider the Lyapunov-like function (6.38). For simplicity and without loss of generality, we assume $\theta_c = 0$. The derivative of V_θ along the solution of (6.37), (6.39) can be computed as

$$\begin{aligned}\dot{V}_\theta &= \epsilon^T\left(A_z - \frac{\dot{m}_s}{m_s}\right)\epsilon + b_z^T\epsilon\tilde{\theta}_u^T\frac{\omega}{m_s} + b_z^T\epsilon\frac{\tilde{x}_2}{m_s} + \tilde{\theta}_u^T\Gamma^{-1}(\dot{\theta}_u - \dot{\hat{\theta}}_u)\\ &\leq -c_0|\epsilon|^2 - \frac{d_0}{2}(b_z^T\epsilon)^2 + \sigma\tilde{\theta}_u^T\hat{\theta}_u + \tilde{\theta}_u^T\Gamma^{-1}\dot{\theta}_u + b_z^T\epsilon\frac{\tilde{x}_2}{m_s}\\ &\leq -c_0|\epsilon|^2 - \frac{d_0}{4}(b_z^T\epsilon)^2 + \sigma\tilde{\theta}_u^T\hat{\theta}_u + \tilde{\theta}_u^T\Gamma^{-1}\dot{\theta}_u + \frac{\tilde{x}_2^2}{d_0 m_s^2}\end{aligned} \tag{6.41}$$

where

$$c_0 = \min_{1\leq i\leq\rho} c_i - \frac{\delta_0}{2}, \quad d_0 = \min_{1\leq i\leq\rho} d_i$$

Using the inequality

$$\sigma\tilde{\theta}_u^T\hat{\theta}_u + \tilde{\theta}_u^T\Gamma^{-1}\dot{\theta}_u \leq -\frac{\sigma_0}{2}|\tilde{\theta}_u|^2 + 2\sigma_0 M^2 + \tilde{\theta}_u^T\Gamma^{-1}\dot{\theta}_u \leq -\frac{\sigma_0}{4}|\tilde{\theta}_u|^2 + 2\sigma_0 M^2 + \frac{1}{2\sigma_0}|\Gamma^{-1}\dot{\theta}_u|^2$$

we obtain

$$\dot{V}_\theta \leq -c_0|\epsilon|^2 - \frac{\sigma_0}{4}|\tilde{\theta}_u|^2 - \frac{d_0}{4}(b_z^T\epsilon)^2 + 2\sigma_0 M^2 + \frac{1}{\sigma_0}|\Gamma^{-1}\dot{\theta}_u|^2 + \frac{\tilde{x}_2^2}{d_0 m_s^2}. \tag{6.42}$$

In view of (6.32), $\frac{\tilde{x}_2}{m_s} \in \mathcal{S}(\mu)\cap\mathcal{L}_\infty$ if $\dot{\theta}_u \in \mathcal{S}(\mu)\cap\mathcal{L}_\infty$, hence (6.42) implies that $\epsilon, \tilde{\theta}_u \in \mathcal{L}_\infty$.

In addition, integrating (6.41), we get that $\forall t, T > 0$,

$$\int_t^{t+T}\left(c_0|\epsilon|^2 + \frac{d_0}{4}(b_z^T\epsilon)^2 - \sigma\tilde{\theta}_u^T\hat{\theta}_u\right)d\tau \leq V_\theta(t) - V_\theta(t+T) + \int_t^{t+T}\left(\tilde{\theta}_u^T\Gamma^{-1}\dot{\theta}_u + \frac{\tilde{x}_2^2}{d_0 m_s^2}\right)d\tau$$

Using $V_\theta, \tilde{\theta}_u \in \mathcal{L}_\infty$ and $\sqrt{|\dot{\theta}_u|}, \frac{\tilde{x}_2}{m_s} \in \mathcal{S}(\mu)$, it follows that $\epsilon, b_z^T\epsilon, \sqrt{-\sigma\tilde{\theta}_u^T\hat{\theta}_u} \in \mathcal{S}(\mu)$, and, consequently, using (i), (ii), it follows that $\sigma\hat{\theta}_u \in \mathcal{S}(\mu)$.

Due to the linearity of the stabilizing functions, b_z depends only on the parameter estimates $\hat{\theta}_u$ and D_z is linear in $\Xi, \Upsilon, \zeta, \varsigma, y$, thus $b_z, \frac{D_z}{m_s} \in \mathcal{L}_\infty$, and $\dot{\hat{\theta}}_u \in \mathcal{L}_\infty\cap\mathcal{S}(\mu)$ follows from (6.39) and $\sigma\hat{\theta}_u \in \mathcal{S}(\mu)$. Using (6.27), $\frac{\phi}{m_s}$ satisfies

$$\frac{d}{dt}\left(\frac{\phi}{m_s}\right) = (A_z - \frac{\dot{m}_s}{m_s}I)\frac{\phi}{m_s} - \frac{D_z}{m_s}\dot{\hat{\theta}}_u \tag{6.43}$$

from which $\frac{\phi}{m_s} \in \mathcal{L}_\infty\cap\mathcal{S}(\mu)$ follows immediately.

If $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{L}_1\cap\mathcal{L}_2\cap\mathcal{L}_\infty$, then $\mu = 0$ and all the $\mathcal{S}(\mu)$ properties become $\mathcal{L}_2$ properties, i.e., $\epsilon, \frac{\phi}{m_s} \in \mathcal{L}_2$. Finally, using (6.37) and (6.43) we see that $\dot{\epsilon}, \frac{d}{dt}\left(\frac{\phi}{m_s}\right) \in \mathcal{L}_\infty$, which together with $\epsilon, \frac{\phi}{m_s} \in \mathcal{L}_2\cap\mathcal{L}_\infty$ implies that $\epsilon, \frac{\phi}{m_s} \to 0$ as $t \to \infty$. ■

Having established the stability properties of the adaptive law, we analyze the closed-loop stability properties of the adaptive control scheme based on the error system (6.21) next. The following theorem summarizes the results of this analysis, which is presented in Appendix B in details.

Theorem 6.1 *The adaptive controller described by (6.20) and (6.39), when applied to the LTV plant (2.1),(2.2), guarantees the existence of a constant $\mu^* > 0$ such that $\forall\mu \in (0,\ \mu^*)$, all the closed-loop signals are uniformly bounded, and the tracking error is of the order μ in the mean square sense, i.e.,*

$$\int_t^{t+T}(y(\tau) - y_r(\tau))^2 d\tau \leq \bar{c}_1\mu T + \bar{c}_2, \quad \forall t, T \geq 0 \tag{6.44}$$

where $\bar{c}_1$ and $\bar{c}_2$ are finite positive constants. Moreover, $\bar{c}_1$ can be expressed as

$$\bar{c}_1 = \frac{\bar{c}}{c_0}\left(\left(1+\frac{1}{d_0}+\|\Gamma^{-1}\|_2\right)\left(1+\frac{\|\Gamma\|_2^2}{d_0^2}\right)+\frac{\|\Gamma\|_2^2}{d_0}\right)\bar{m} \tag{6.45}$$

where $\bar{m} \triangleq \|m_s\|_\infty$ and $\bar{c}$ is a finite positive constant independent of c_0, d_0, Γ.

Proof: The proof is similar to that of Theorem 5.1 and is presented in Appendix B. ■

Theorem 6.1 indicates that, using the adaptive controller (6.20), (6.39), only the time variations of the unstructured plant parameters are required to be small in the mean square sense to guarantee closed-loop stability and tracking with small MSE. The overall system is not necessarily restricted to be slowly TV. The requirement about the time variations of the unstructured plant parameters is necessary since it is not possible to estimate unknown and arbitrarily fast TV parameters using a general adaptive law with finite speed of adaptation [22, 28]. Once this requirement is satisfied, the mean square tracking error is guaranteed to be of the order of the speed of the unstructured plant parameter variations.

Besides establishing stability and tracking properties, the theorem provides guidelines for performance improvement as well. It shows that the MSE performance can be improved by amplifying c_0, d_0, and possibly Γ for μ small enough to satisfy the stability conditions. Unlike [12], arbitrary performance improvement is only assured in terms of the normalized tracking error $\frac{y-y_r}{m_s}$. The bound on $MSE(y-y_r)$ depends on $\bar{m} = \|m_s\|_\infty$. However, although $\bar{m}$ might increase by increasing c_0, d_0, $\|\Gamma\|_2$, this can be counteracted by reducing the normalization coefficient δ_1. Hence the performance of the adaptive backstepping controller can be improved by adjusting the design parameters c_0, d_0, $\|\Gamma\|_2$, and δ_1. We demonstrate this fact via simulations in Section 7.

6.3 Fully Structured Parameter Variations

The case of fully structured parameter variations corresponds to θ_u being constant. We generalize it to the situation that $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_\infty$. For this special class of LTV plant, the proposed adaptive controller (6.39), (6.20) has the following properties:

Corollary 6.1 *If the speed of parameter variations satisfy $\sqrt{|\dot{\theta}_u|}, \dot{\theta}_u \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_\infty$, then the adaptive controller (6.20), (6.39) guarantees that all the closed-loop signals are uniformly bounded, and the tracking error converges to zero asymptotically.*

Proof: This is a direct consequence of Theorem 6.1 and Lemma 6.2 (iii). ■

Due to the transformation (6.1), the parameter vector θ_u may not reflect the plant parameters themselves, and can contain more or less than $n+m+1$ elements, which corresponds to the overparameterized and the underparameterized case, respectively. Corollary 6.1 indicates that when full knowledge of the parameter variations is available, then regardless of the speed of the parameter variations of the plant, global stability is guaranteed, and asymptotic tracking is achieved.

In addition, in the case of fully structured parameter variations, $\tilde{x}_2$ is exponentially vanishing. Therefore, in this case, we can apply the tuning design given in [6, 17] instead of the certainty equivalence approach using parameterization (6.10). The advantage is a guaranteed performance improvement, as in the TI case [7, 12].

7 Simulation Results

Let us consider a simple unstable second order LTV plant whose state space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} g_1^*(t)\sin(10t) + g_2^*(t) & 1 \\ g_3^*(t)\cos(3t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{7.1}$$

$$y = x_1 \tag{7.2}$$

where $g_1^*(t) = 1$, $g_2^*(t) = 2 + \sin(0.1t)$, and $g_3^*(t) = 2$. It is required to design a controller so that the output y tracks the reference signal $y_r(t) = \sin(t)$.

Let us first apply the pointwise design, i.e., the control scheme (3.18)-(3.26) together with the estimation filters (3.8),(3.9), assuming that g_1^*, g_2^*, g_3^* are all known exactly. Noting that $n = 2, m = 0, \rho = 2$ for the plant, the filter parameters are chosen as $k_1 = k_2 = 4$, and the design parameters are chosen as $c_1 = c_2 = d_1 = d_2 = 1$. Figure 1 (a) shows the result. Although the output signal is bounded, tracking performance is not that successful. Next, we increase the values of the design constants c_1 and c_2 to 5. Tracking performance is enhanced as shown in Figure 1 (b). However, asymptotic tracking is not achieved.

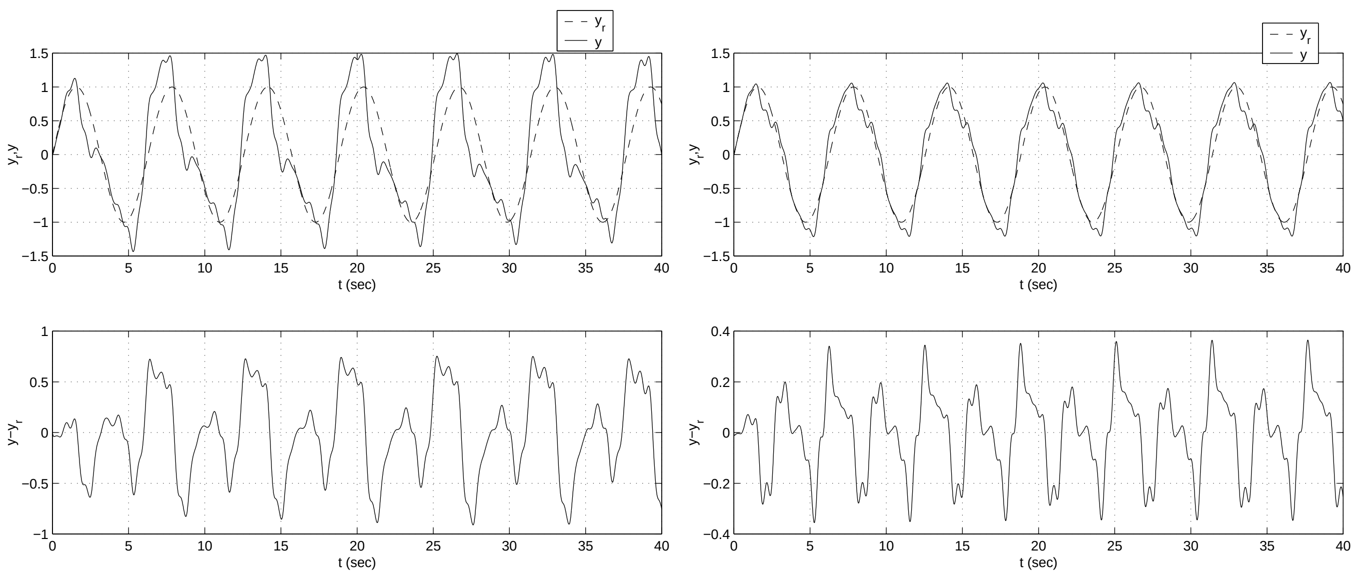

Figure 1: Response using the pointwise design and exact knowledge of g_1^*, g_2^*, g_3^*: (a) $c_1 = d_1 = c_2 = d_2 = 1$. (b) $c_1 = c_2 = 5$, $d_1 = d_2 = 1$.

Then, we repeat the same simulations with the LTV design i.e., the control scheme (4.12)-(4.16) together with the estimation filters (4.6)-(4.8). Tracking is perfect as shown in Figure 2. Next, we consider some parametric uncertainty. We assume that our plant model is a little bit erroneous, e.g., models of the actual functions g_1^*, g_2^*, g_3^* of the plant are $g_1(t) = 1, g_2(t) = g_3(t) = 2$. Choosing the design parameters as $c_1 = c_2 = 5, d_1 = d_2 = 1$, we can see from Figure 3 that the system is stabilized, and a relatively small tracking error (smaller than that of controller (3.26) with known parameters) is obtained. Note that the parametric uncertainty has amplitude 1, however its derivative has a much smaller amplitude 0.1. This demonstrates that the time variation of the uncertain parameter, not the size of uncertainty itself, determines the system stability and performance.

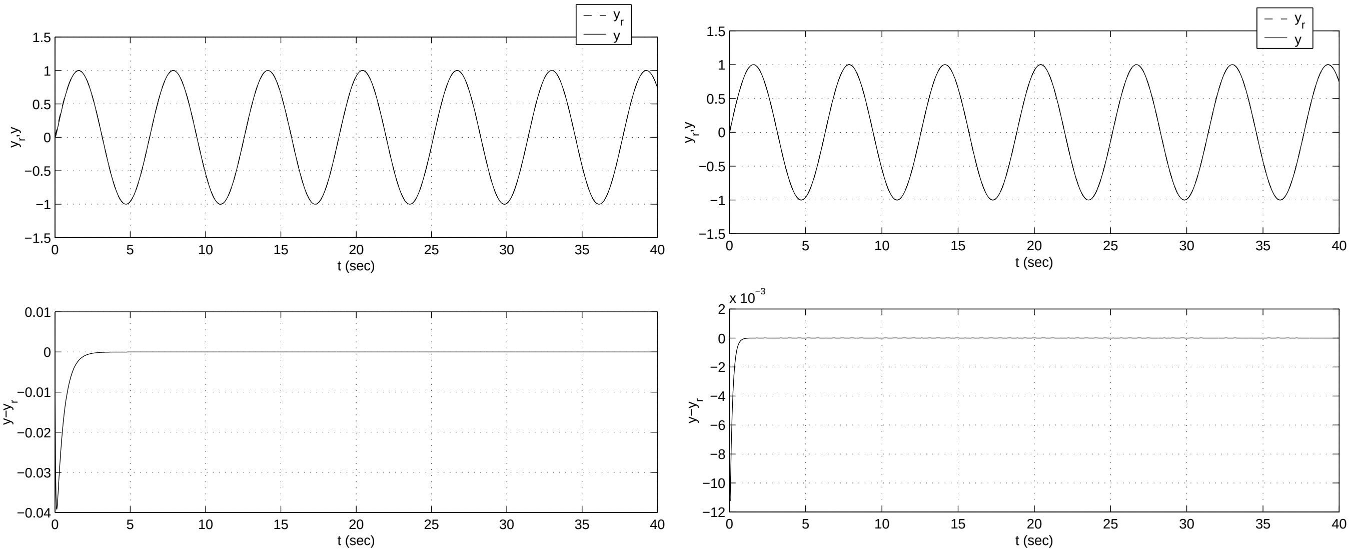

Figure 2: Response using the LTV design and exact knowledge of g_1^*, g_2^*, g_3^*: (a) $c_1 = d_1 = c_2 = d_2 = 1$. (b) $c_1 = c_2 = 5$, $d_1 = d_2 = 1$.

Finally, we consider the unknown parameter case assuming that the plant structure (7.1)-(7.2) is known but the functions g_1^*, g_2^*, g_3^* are unknown. In order to build up an adaptive controller, we first write the plant parameter in structured parameter variations form as follows:

$$a(t) = \begin{bmatrix} -g_1^*(t)\sin(10t) - g_2^*(t) \\ -g_3^*(t)\cos(3t) \end{bmatrix} = -\Psi_a(t)\theta_u(t) - \theta_{0a}(t), \quad b(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \Psi_b(t)\theta_u(t) + \theta_{0b}(t)$$

where

$$\theta_u(t) = \begin{bmatrix} g_1^*(t) \\ g_2^*(t) \\ g_3^*(t) \end{bmatrix}, \quad \Psi_a(t) = \begin{bmatrix} \sin(10t) & 1 & 0 \\ 0 & 0 & \cos(3t) \end{bmatrix}, \quad \Psi_b(t) = 0, \quad \theta_{0a}(t) = 0, \quad \theta_{0b}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Noting that Ψ_b and hence Υ are zero, the estimation filters are implemented using equations (6.3), (6.5), and (6.6). The following control law is designed based on the steps in Section 6 selecting $\delta_u = 0$:

$$\begin{aligned} u &= -z_1 - c_2 z_2 - d_2 s^2 z_2 - \kappa_2(\hat{\theta}_u^T \omega + \zeta_2 + \varsigma_2) + k_2(\zeta_1 + \varsigma_1 - y + \hat{\theta}_u^T \Xi_1) - \hat{\theta}_u^T \Psi_{a2} y + (c_1 + d_1)\cos t - \sin t \\ z_1 &= y - y_r \\ z_2 &= (c_1 + d_1) z_1 + \hat{\theta}_u^T \omega + \zeta_2 + \varsigma_2 - \dot{y}_r \\ \kappa_2 &= c_1 + d_1 + \hat{\theta}_u^T \Psi_{a1} \\ s^2 &= \kappa^2 + 4 + \sin^2(10t) \\ \omega &= \Psi_{a1} y + \Xi_2 \end{aligned}$$

The adaptive law and the associated auxiliary signal are defined as

$$\dot{\hat{\theta}}_u = \Gamma [\, 1 \quad \kappa \,] \epsilon \frac{\omega}{m_s} - \Gamma \sigma \hat{\theta}_u \tag{7.3}$$

$$\epsilon = \frac{1}{m_s}(z + \phi), \quad z = [\, z_1 \quad z_2 \,]^T \tag{7.4}$$

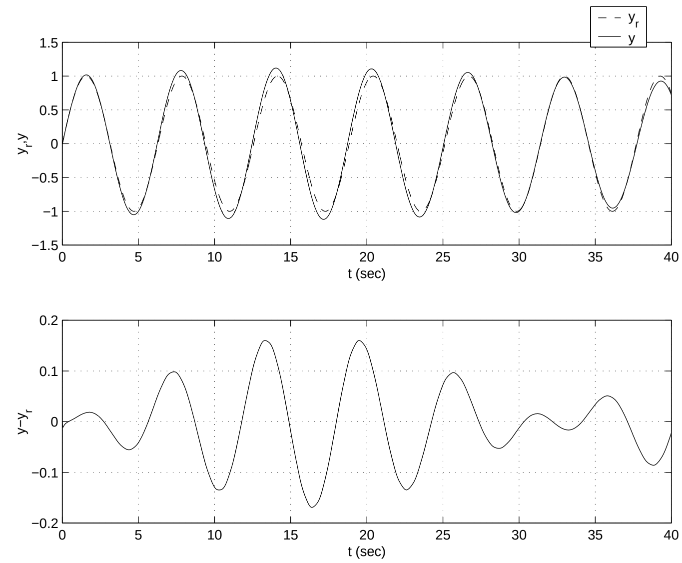

Figure 3: Response using the LTV design in presence of parametric uncertainty ($g_1(t) = 1, g_2(t) = g_3(t) = 2$, $c_1 = c_2 = 5$, $d_1 = d_2 = 1$).

$$\dot{n}_s = -\delta_0 n_s + u^2 + y^2, \quad m_s = \sqrt{1 + \delta_1 n_s} \tag{7.5}$$

$$\dot{\phi} = \begin{bmatrix} -c_1 - d_1 & 1 \\ -1 & -c_2 - d_2 s^2 \end{bmatrix} \phi - \begin{bmatrix} 0 \\ \omega^T \dot{\hat{\theta}}_u \end{bmatrix} \tag{7.6}$$

where c_1, d_1, c_2, d_2, Γ, δ_0, δ_1 are design constants, σ is the switching-σ coefficient defined in (6.40).

Figure 4 shows the tracking error $z_1 = y - y_r$ and the parameter estimate $\hat{\theta}_u$ for simulations with different choices of the design parameters and the adaptive gain Γ parameters. The switching and normalization parameters are set as $\sigma_0 = 0.1$, $M = 10$, $\delta_0 = 2$, $\delta_1 = 1$ in all of these simulations. The response for $\Gamma = 10, c_1 = c_2 = 5, d_1 = d_2 = 1$ is redrawn in Figure 5(a) in order to make it comparable with the results for the cases with known and unknown parameters. As can be seen in these figures, the system is stabilized, and the tracking error remains in a neighborhood of 0, for all design parameter choices.

As can be seen in Figure 4, increasing the value of the design parameters $c_1 = c_2$ improves the tracking performance as in the known parameter case. By increasing the adaptive gain, not only parameter estimation gets faster but the tracking performance is improved further as well.

Later, fixing $\Gamma = 10, c_1 = c_2 = 5, d_1 = d_2 = 1, \sigma_0 = 0.1, M = 10, \delta_0 = 2$, the effect of the normalization coefficient δ_1 is tested. The results in terms of the tracking errors are shown in Figure 5(b). As seen in this figure decreasing δ_1 has a similar effect with increasing Γ on enhancement of tracking.

In the simulations above, we see that the parameter estimates adapt to the parameter changes. We have also observed that the control effort remains within a reasonable bound. Since the only unknown TV parameter $a(t)$ is slowly time varying, stability is guaranteed.

8 Conclusions

In this paper, we introduced a new backstepping controller for LTV systems with known and unknown parameters. The controller guarantees exponential tracking when the plant parameters are known exactly.

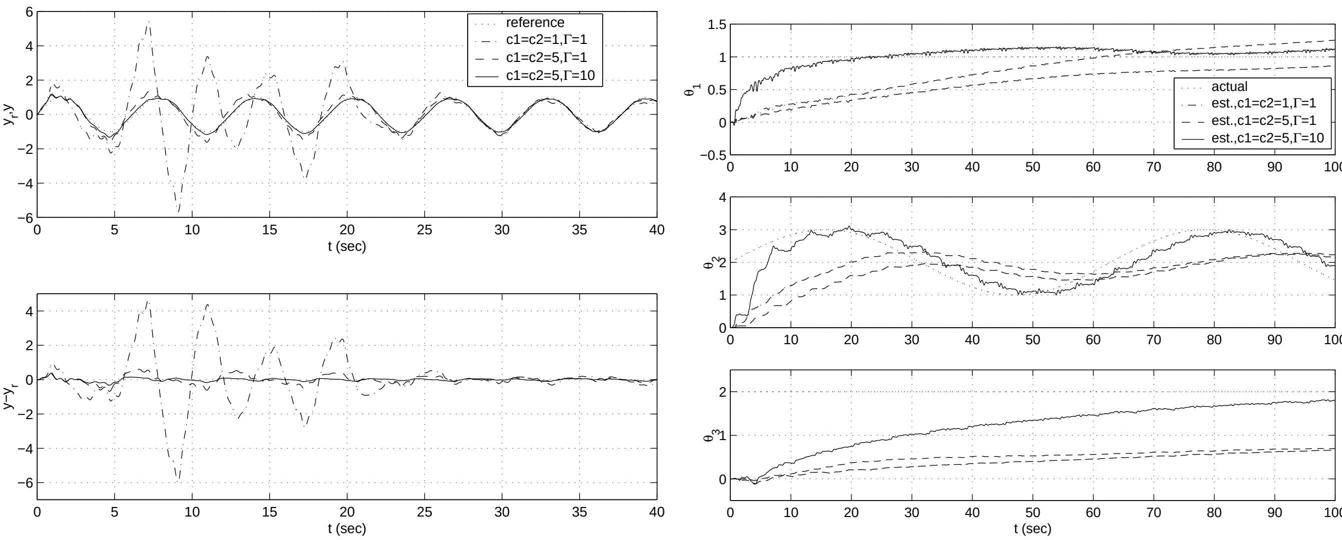

Figure 4: Adaptive control with different choices for c_1, c_2, Γ ($d_1 = d_2 = 1, \delta_0 = 2, \delta_1 = 1$): (a) Tracking. (b) Parameter estimation.

When the plant parameters are not known exactly but their time variations are small enough, regardless of the size of the parameter errors (except for the high frequency gain), global stability can be guaranteed by choosing certain design parameters properly. Hence, the proposed controller has strong parametric robustness properties which most of the traditional model reference controllers do not have.

When the plant parameters are unknown, the proposed controller is combined with an online parameter estimator to form a new adaptive controller. This new adaptive controller guarantees the following: All the closed loop signals are globally uniformly bounded. The tracking error remains small and of the order of the speed of the unstructured plant parameter variations, which is required to be small in the mean square sense. If the plant parameter variations are fully structured, the tracking error converges asymptotically to zero. The performance bounds for the tracking error developed can be used to select certain design parameters for performance improvement.

Appendix A: Proof of Theorem 5.1

Using (5.3), (4.12)–(4.15), we obtain the following error equation:

$$\dot{z} = A_z(z,\theta,t)z + b_z(z,\theta,t)(-\tilde{\theta}^T\omega + \eta_y + \eta_u + \epsilon_t) \tag{A.1}$$

where

$$z = \begin{bmatrix} z_1 & z_2 & \cdots & z_\rho \end{bmatrix}^T \tag{A.2}$$

$$A_z = \begin{bmatrix} -c_1 - d_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - d_2\kappa_2^2 & 1 & \ddots & \vdots \\ 0 & -1 & -c_3 - d_3\kappa_3^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -c_\rho - d_\rho\kappa_\rho^2 \end{bmatrix} \tag{A.3}$$

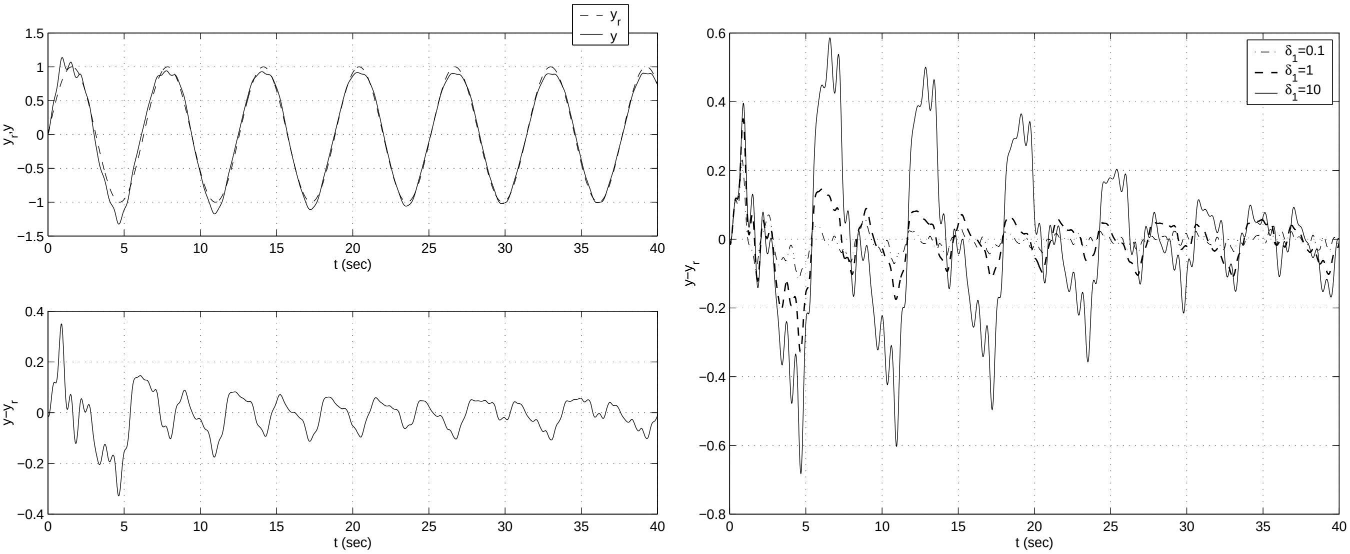

Figure 5: (a) Response using the adaptive controller ($c_1 = c_2 = 5, d_1 = d_2 = 1, \Gamma = 10, \delta_0 = 2, \delta_1 = 1$). (b) Adaptive tracking with different choices for δ_1 ($c_1 = c_2 = 5, d_1 = d_2 = 1, \Gamma = 10, \delta_0 = 2$).

$$b_z = [1 \quad \kappa_2 \quad \kappa_3 \quad \cdots \quad \kappa_\rho]^T \tag{A.4}$$

$$\kappa_i = -\frac{\partial \alpha_{i-1}}{\partial y}, \quad i = 2, \cdots, \rho \tag{A.5}$$

We first consider the term $\tilde{\theta}^T \omega$. We have

$$\begin{aligned}
\tilde{\theta}^T \omega &= -\sum_{i=0}^{n-1} \tilde{a}_i \frac{s^i(s+k_1)}{K(s)} y + \sum_{i=0}^{m} \tilde{b}_i \frac{s^i(s+k_1)}{K(s)} u \\
&= [-\sum_{i=0}^{n-1} \tilde{a}_i \frac{s^i(s+k_1)}{K(s)} + \sum_{i=0}^{m} \tilde{b}_i s^i (s+k_1)(Z_l(s,t)K(s))^{-1} R_l(s,t) - \tilde{b}_m s] y + \tilde{b}_m \dot{y} \\
&= \{-\sum_{i=0}^{n-1} \tilde{a}_i s^i (s+k_1) K(s)^{-1} + \sum_{i=0}^{m-1} \tilde{b}_i s^i (s+k_1)(Z_l(s,t)K(s))^{-1} R_l(s,t) \\
&\quad + \tilde{b}_m s[(s^{m-1}(s+k_1))(Z_l(s,t)K(s))^{-1} R_l(s,t) - 1]\} y + \tilde{b}_m \dot{y}
\end{aligned} \tag{A.6}$$

Let the ρth order monic polynomial $K_\rho(s)$ and the mth order monic polynomial $K_m(s)$ be a decomposition of $K(s)$, i.e., $K(s) = K_\rho(s) K_m(s)$. Then the operator in $[\cdot]$ of (A.6) can be written as

$$\begin{aligned}
& s^{m-1}(s+k_1)[Z_l(s,t)K(s)]^{-1} R_l(s,t) - K_m(s)[Z_l(s,t)K(s)]^{-1} R_l(s,t) \\
& + K_m(s)[Z_l(s,t)K(s)]^{-1} R_l(s,t) - K_m(s)[Z_l(s,t)K(s)]^{-1} Z_l(s,t) K_\rho(s) \\
= \; & (s^{m-1}(s+k_1) - K_m(s))[Z_l(s,t)K(s)]^{-1} R_l(s,t) \\
& + K_m(s)[Z_l(s,t)K(s)]^{-1}(R_l(s,t) - Z_l(s,t)K_\rho(s))
\end{aligned}$$

which is the sum of two proper and exponentially stable I/O operators. Hence the operator in $\{\cdot\}$ of (A.6), which we denote as $Q(s,t)$, is a proper and exponentially stable I/O operator. Using (A.6), (4.12), and the definition of $Q(s,t)$, we can write

$$\tilde{\theta}^T \omega = Q(s,t) y + \tilde{b}_m \dot{y} = Q(s,t) z_1 + \tilde{b}_m \dot{z}_1 + \eta_d \tag{A.7}$$

where $\eta_d = Q(s,t)y_r + \tilde{b}_m \dot{y}_r \in \mathcal{L}_\infty$. Suppose that $\{A_q(t), b_q(t), c_q(t), d_q(t))$ is a state space realization of $Q(s,t)$, then

$$\begin{aligned} \dot{q} &= A_q q + b_q z_1 && \text{(A.8)} \\ \tilde{\theta}^T \omega &= c_q^T q + d_q z_1 + \tilde{b}_m \dot{z}_1 + \eta_d && \text{(A.9)} \end{aligned}$$

Note that $A_q, b_q, c_q, d_q \in \mathcal{L}_\infty$ and are independent of the design parameters c_i, d_i. Since $A_q(t)$ is an exponentially stable matrix, there exists a positive definite matrix $P_q(t) \in \mathcal{L}_\infty$ such that

$$\dot{P}_q(t) + A_q(t)^T P_q(t) + P_q(t) A_q(t) = -2I \tag{A.10}$$

Define $\kappa_1^2 \stackrel{\triangle}{=} \frac{1}{1-\tilde{b}_m}$, then we substitute (A.9) into the error equation of z_1 and get

$$\dot{z}_1 = -(c_1 + d_1)\kappa_1^2 z_1 + \kappa_1^2 z_2 - \kappa_1^2 c_q^T q - \kappa_1^2 d_q z_1 - \kappa_1^2 \eta_d + \kappa_1^2 \eta_y + \kappa_1^2 \eta_u + \kappa_1^2 \epsilon_t \tag{A.11}$$

Next, we consider the term η_y, which we write in the following state space form

$$\begin{aligned} \dot{\chi} &= A_0 \chi - \textstyle\sum_{i=0}^{n-1} \dot{\hat{a}}_i \xi_i \\ \eta_y &= \chi_2 \end{aligned} \tag{A.12}$$

Augmenting (A.12) with (3.8), we get

$$\begin{bmatrix} \dot{\chi} \\ \dot{\xi}_0 \end{bmatrix} = \begin{bmatrix} A_0 & -\sum_{i=0}^{n-1} \dot{\hat{a}}_i A_0^i \\ 0 & A_0 \end{bmatrix} \begin{bmatrix} \chi \\ \xi_0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_n \end{bmatrix} y \tag{A.13}$$

Finally, we augment the error systems (A.1) and (A.8) with (A.13) using (A.11) and get

$$\begin{bmatrix} \dot{z} \\ \dot{q} \\ \dot{\chi} \\ \dot{\xi}_0 \end{bmatrix} = \begin{bmatrix} \hat{A}_z - \hat{b}_z d_q e_1^T & -\hat{b}_z c_q^T & \hat{b}_z e_2^T & 0 \\ b_q e_1^T & A_q & 0 & 0 \\ 0 & 0 & A_0 & -\sum_{i=0}^{n-1} \dot{\hat{a}}_i A_0^i \\ e_n e_1^T & 0 & 0 & A_0 \end{bmatrix} \begin{bmatrix} z \\ q \\ \chi \\ \xi_0 \end{bmatrix} + \begin{bmatrix} \hat{b}_z(-\eta_d + \epsilon_t) \\ 0 \\ \vdots \\ y_r \end{bmatrix} + \begin{bmatrix} \hat{b}_z \eta_u \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A.14}$$

where

$$\hat{A}_z = \begin{bmatrix} -c_1\kappa_1^2 - d_1\kappa_1^2 & \kappa_1^2 & 0 & \cdots & \cdots & \cdots & 0 \\ \kappa_2(\kappa_1^2-1)(c_1+d_1) - 1 & -\kappa_2(\kappa_1^2-1) - c_2 - d_2\kappa_2^2 & 1 & \ddots & & & \vdots \\ \kappa_3(\kappa_1^2-1)(c_1+d_1) & -\kappa_3(\kappa_1^2-1) - 1 & -c_3 - d_3\kappa_3^2 & \ddots & \ddots & & \vdots \\ \kappa_4(\kappa_1^2-1)(c_1+d_1) & -\kappa_4(\kappa_1^2-1) & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ \kappa_\rho(\kappa_1^2-1)(c_1+d_1) & -\kappa_\rho(\kappa_1^2-1) & 0 & \cdots & 0 & -1 & -c_\rho - d_\rho\kappa_\rho^2 \end{bmatrix} \tag{A.15}$$

$$\hat{b}_z = \begin{bmatrix} \kappa_1^2 & (2-\kappa_1^2)\kappa_2 & (2-\kappa_1^2)\kappa_3 & \cdots & (2-\kappa_1^2)\kappa_\rho \end{bmatrix}^T \tag{A.16}$$

We first analyze the homogeneous part of the system (A.14) by individually considering the following two partial Lyapunov functions

$$\begin{aligned} V_1 &= \frac{\lambda_1}{2} q^T P_q q + \frac{1}{2}\sum_{i=1}^{\rho} z_i^2 && \text{(A.17)} \\ V_2 &= \frac{1}{2}\chi^T P_0 \chi + \frac{\lambda_2}{2}\xi_0^T P_0 \xi_0 && \text{(A.18)} \end{aligned}$$

where $\lambda_1, \lambda_2 > 0$ are constants to be chosen and $P_0 = P_0^T > 0$ is a constant matrix satisfying $A_0^T P_0 + P_0 A_0 = -2I$. The derivatives of V_1, V_2 along the solution of (A.14) are computed as

$$\begin{aligned}
\dot{V}_1 &= -\lambda_1|q|^2 + \lambda_1 b_q^T P_q q z_1 - \kappa_1^2 c_1 z_1^2 - \sum_{i=2}^{\rho} c_i z_i^2 - \kappa_2(\kappa_1^2 - 1)z_2^2 - \sum_{i=1}^{\rho} d_i \kappa_i^2 z_i^2 - \hat{b}_z^T z d_q z_1 \\
&\quad + (\kappa_1^2 - 1)z_1 z_2 + \sum_{i=2}^{\rho} \kappa_i(\kappa_1^2 - 1)(c_1 + d_1)z_1 z_i - \sum_{i=3}^{\rho} \kappa_i(\kappa_1^2 - 1)z_2 z_i \\
&\quad - \hat{b}_z^T z c_q^T q + \hat{b}_z^T z e_2^T \chi
\end{aligned} \tag{A.19}$$

$$\dot{V}_2 = -|\chi|^2 - \chi^T P_0 \sum_{i=0}^{n-1} \dot{\hat{a}}_i A_0^i \xi_0 + \lambda_2 \xi_0^T P_0 e_n z_1 - \lambda_2 |\xi_0|^2 \tag{A.20}$$

Using the inequalities

$$\begin{aligned}
\frac{1}{\beta_1} \leq \; &\kappa_1^2 \leq \frac{1}{\beta_0} \\
|\kappa_1^2 - 1| &\leq \bar{\beta}_1 \triangleq \max\left(\frac{1}{\beta_0} - 1, 1 - \frac{1}{\beta_1}\right) \\
|2 - \kappa_1^2| &\leq \bar{\beta}_2 \triangleq \max\left(\frac{1}{\beta_0} - 2, 2 - \frac{1}{\beta_1}\right) \\
\lambda_1 b_q^T P_q q z_1 &\leq \frac{\lambda_1}{2}|q|^2 + \frac{\lambda_1 \|P_q b_q\|_\infty^2}{2}|z_1|^2 \\
-\kappa_2(\kappa_1^2 - 1)z_2^2 &\leq \frac{d_2}{8}\kappa_2^2 z_2^2 + \frac{2}{d_2}\bar{\beta}_1^2 z_2^2 \\
-\kappa_1^2 d_q z_1^2 &\leq \frac{d_1}{4}\kappa_1^2 z_1^2 + \frac{1}{d_1 \beta_0}\|d_q\|_\infty^2 z_1^2 \\
-\kappa_i(2 - \kappa_1^2)d_q z_1 z_i &\leq \frac{d_i}{8}\kappa_i^2 z_i^2 + \frac{2}{d_i}\bar{\beta}_2^2 \|d_q\|_\infty^2 z_1^2 \\
(\kappa_1^2 - 1)z_1 z_2 &\leq \frac{c_2}{2} z_2^2 + \frac{\bar{\beta}_1^2}{2c_2} z_1^2 \\
\kappa_i(\kappa_1^2 - 1)(c_1 + d_1)z_1 z_i &\leq \frac{d_i}{4}\kappa_i^2 z_i^2 + \frac{1}{d_i}\bar{\beta}_1^2 (c_1 + d_1)^2 z_1^2 \\
-\kappa_i(\kappa_1^2 - 1)z_2 z_i &\leq \frac{d_i}{8}\kappa_i^2 z_i^2 + \frac{2}{d_i}\bar{\beta}_1^2 z_2^2 \\
-\kappa_1^2 z_1 c_q^T q &\leq \frac{d_1}{4}\kappa_1^2 z_1^2 + \frac{1}{d_1 \beta_0}\|c_q\|_\infty^2 |q|^2 \\
-\kappa_i(2 - \kappa_1^2) z_i c_q^T q &\leq \frac{d_i}{8}\kappa_i^2 z_i^2 + \frac{2}{d_i}\bar{\beta}_2^2 \|c_q\|_\infty^2 |q|^2 \\
\kappa_1^2 z_1 e_2^T \chi &\leq \frac{d_1}{4}\kappa_1^2 z_1^2 + \frac{1}{d_1 \beta_0}|\chi|^2 \\
\kappa_i(2 - \kappa_1^2) z_i e_2^T \chi &\leq \frac{d_i}{8}\kappa_i^2 z_i^2 + \frac{2}{d_i}\bar{\beta}_2^2 |\chi|^2 \\
-\chi^T P_0 \sum_{i=0}^{n-1} \dot{\hat{a}}_i A_0^i \xi_0 &\leq \frac{1}{2}|\chi|^2 + \frac{1}{2}\|P_0 \sum_{i=0}^{n-1} \dot{\hat{a}}_i A_0^i\|_\infty^2 |\xi_0|^2 \\
\xi_0^T P_0 e_n z_1 &\leq \frac{1}{2}\|P_0\|^2 z_1^2 + \frac{1}{2}|\xi_0|^2
\end{aligned}$$

we get

$$
\begin{aligned}
\dot{V}_1 &\leq -\frac{\lambda_1}{2}|q|^2 - \left(\kappa_1^2 c_1 - \frac{\bar{\beta}_1^2}{2c_2} - \frac{\lambda_1\|P_0 b_q\|_\infty^2}{2}\right) z_1^2 - \frac{c_2}{2}z_2^2 - \sum_{i=3}^{\rho} c_i z_i^2 - \sum_{i=1}^{\rho}\frac{d_i}{4}\kappa_i^2 z_i^2 \\
&\quad + \left(\frac{1}{d_1\beta_0} + \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_2^2\right)(\|c_q\|_\infty^2|q|^2 + \|d_q\|_\infty^2 z_1^2 + |\chi|^2) \\
&\quad + \sum_{i=2}^{\rho}\frac{1}{d_i}\bar{\beta}_1^2(c_1+d_1)^2 z_1^2 + \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_1^2 z_2^2 \qquad \text{(A.21)} \\
\dot{V}_2 &\leq -\frac{1}{2}|\chi|^2 - \left(\frac{\lambda_2}{2} - \frac{1}{2}\|P_0\sum_{i=0}^{n-1}\dot{\hat{a}}_i A_0^i\|_\infty^2\right)|\xi_0|^2 + \frac{\lambda_2}{2}\|P_0\|^2 z_1^2 \qquad \text{(A.22)}
\end{aligned}
$$

Let us choose

$$
\lambda_2 \geq \|P_0\sum_{i=0}^{n-1}\dot{\hat{a}}_i A_0^i\|_\infty^2 + 2\lambda_{20}
$$

where $\lambda_{20} > 0$ is an arbitrary constant, and consider the following Lyapunov function

$$
V = V_1 + \lambda_3 V_2 \qquad \text{(A.23)}
$$

where $\lambda_3 > 0$ is another constant to be chosen. Using (A.21), (A.22), we have the following

$$
\begin{aligned}
\dot{V} &\leq -\lambda_3\lambda_{20}|\xi_0|^2 - \left[\frac{\lambda_1}{2} - \left(\frac{1}{d_1\beta_0} + \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_2^2\right)\|c_q\|_\infty^2\right]|q|^2 - \left(\frac{\lambda_3}{2} - \frac{1}{d_1\beta_0} - \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_2^2\right)|\chi|^2 \\
&\quad - \left[\kappa_1^2 c_1 - \frac{\bar{\beta}_1^2}{2c_2} - \frac{\lambda_1\|P_q b_q\|_\infty^2}{2} - \frac{\|d_q\|_\infty^2}{d_1\beta_0} - \sum_{i=2}^{\rho}\frac{1}{d_i}\bar{\beta}_2^2\left(2\|d_q\|_\infty^2 + (c_1+d_1)^2\right) - \frac{\lambda_2\lambda_3}{2}\right] z_1^2 \\
&\quad - \left(\frac{c_2}{2} - \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_1^2\right) z_2^2 - \sum_{i=3}^{\rho} c_i z_i^2 - \sum_{i=1}^{\rho}\frac{d_i}{4}\kappa_i^2 z_i^2 \qquad \text{(A.24)}
\end{aligned}
$$

We first pick

$$
\frac{\lambda_1}{2} \geq \left(\frac{1}{d_1\beta_0} + \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_2^2\right)\|c_q\|_\infty^2 + \lambda_{10}, \quad \frac{\lambda_3}{2} \geq \frac{1}{d_1\beta_0} + \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_2^2 + \lambda_{30} \qquad \text{(A.25)}
$$

where λ_{10}, $\lambda_{30} > 0$ are arbitrary constants, then if

$$
\begin{aligned}
\frac{c_2}{2} &\geq \sum_{i=2}^{\rho}\frac{2}{d_i}\bar{\beta}_1^2 + c_{20}, \qquad \text{(A.26)} \\
c_1 &\geq \beta_1\left(\frac{\bar{\beta}_1^2}{2c_2} + \frac{\lambda_1\|P_q b_q\|_\infty^2}{2} + \frac{\|d_q\|_\infty^2}{d_1\beta_0} + \bar{\beta}_2^2 + \frac{\lambda_2\lambda_3}{2} + c_{10}\right), \qquad \text{(A.27)} \\
d_i &\geq (\rho-1)\left(2\|d_q\|_\infty^2 + (c_1+d_1)^2\right) \quad \text{for } i = 2, \cdots, \rho, \qquad \text{(A.28)}
\end{aligned}
$$

where $c_{10}, c_{20} > 0$ are arbitrary constants, then

$$
\dot{V} \leq -\lambda_3\lambda_{20}|\xi_0|^2 - \lambda_{10}|q|^2 - \lambda_{30}|\chi|^2 - c_0|z|^2 - \sum_{i=1}^{\rho}\frac{d_i}{4}\kappa_i^2 z_i^2 \qquad \text{(A.29)}
$$

where $c_0 = \min\{c_{10}, c_{20}, c_3, \cdots, c_\rho\}$. Since λ_{10}, λ_{20}, λ_{30}, c_{10}, c_{20} are arbitrary, the existence of λ_1, λ_2, λ_3 is guaranteed provided that (5.4), (5.5), and (5.6) are satisfied. Hence, if c_1, c_2, and d_i for $i = 2, \cdots, \rho$ are chosen to satisfy (5.4), (5.5), and (5.6), then the homogeneous part of (A.14) is exponentially stable. Now let us suppose c_1, c_2, and d_i for $i = 2, \cdots, \rho$ are chosen as such, and go back to equation (A.14). Since $\eta_d, \epsilon_t, y_r \in \mathcal{L}_\infty$, the corresponding output is also bounded. Therefore it suffices to consider the subsystem where η_d, ϵ_t, y_r are all zero.

Next, we define a fictitious normalizing signal

$$m_f = \sqrt{1 + \|(u)_t\|_{2\delta}^2} \tag{A.30}$$

where $0 < \delta < 2\max\{\lambda_3\lambda_{20}, \lambda_{10}, \lambda_{30}, c_0\}$ is a constant such that $K(s - \frac{\delta}{2})^{-1}$, $Z_l(t, s - \frac{\delta}{2})^{-1}$ are exponentially stable. The normalizing property of m_f can be described by the following lemma.

Lemma A.1 *Regardless of the boundedness of any closed loop signal, we have*

$$\frac{|\eta_u|}{m_f} \le c_b\Delta, \ \ \textit{where } \Delta = \sum_{i=0}^{m} \|(\dot{\tilde{b}}_i)_t\|_{p_i\delta^*} \tag{A.31}$$

for all $p_i \ge 1$ and some $\delta^*, c_b > 0$.

Proof: Using the inequality

$$m_f(t) \ge e^{-\frac{\delta}{2}(t-\tau)} m_f(\tau), \ \forall t \ge \tau \ge 0, \tag{A.32}$$

which follows directly from the definition of m_f given by (A.30), we have that for all $\delta_1 > \frac{\delta}{2}$ for which $K(s - \delta_1)^{-1}$ is exponentially stable,

$$\begin{aligned}
\frac{|W_2(s)[\sum_{i=0}^m \dot{\tilde{b}}_i v_i]|}{m_f} &\le \int_0^t |w_{2,\delta_1}(t-\tau)| e^{-\delta_1(t-\tau)} \left| \sum_{i=0}^{m} |\dot{\tilde{b}}_i(\tau)| \frac{|v_i(\tau)|}{m_f(t)} \right| d\tau \\
&\le |w_{2,\delta_1}|_\infty \int_0^t e^{-(\delta_1 - \frac{\delta}{2})(t-\tau)} \sum_{i=0}^{m} |\dot{\tilde{b}}_i(\tau)| \frac{|v_i(\tau)|}{m_f(\tau)} d\tau \\
&\le |w_{2,\delta_1}|_\infty \sum_{i=0}^{m} \|(sI - A_0)^{-1} e_{n-i}\|_{2\delta} \int_0^t e^{-\delta^*(t-\tau)} \sum_{i=0}^{m} |\dot{\tilde{b}}_i(\tau)| d\tau
\end{aligned} \tag{A.33}$$

where $w_{2,\delta_1}(t)$ is the impulse response of $W_2(s - \delta_1)$ and $\delta^* = \delta_1 - \frac{\delta}{2}$. Note that $|w_{2,\delta_1}|_\infty$ is finite since $K(s - \delta_1)^{-1}$ is exponentially stable. Using Holder's inequality, we have that $\forall p > 1$ and $\bar{p} = (1 - \frac{1}{p})$,

$$\int_0^t e^{-\delta^*(t-\tau)} |\dot{\tilde{b}}_i(\tau)| d\tau \le \left(\int_0^t e^{-\delta^*(t-\tau)} d\tau \right)^{\frac{1}{\bar{p}}} \left(\int_0^t e^{-\delta^*(t-\tau)} |\dot{\tilde{b}}_i(\tau)|^p d\tau \right)^{\frac{1}{p}} \le \frac{1}{(\delta^*)^{\frac{1}{\bar{p}}}} \|(\dot{\tilde{b}})_t\|_{p\delta^*} \tag{A.34}$$

The result follows directly from (A.33) and (A.34). ■

Now define the following normalized errors

$$\bar{z} = \frac{z}{m_f}, \quad \bar{q} = \frac{q}{m_f}, \quad \bar{\chi} = \frac{\chi}{m_f}, \quad \bar{\xi}_0 = \frac{\xi_0}{m_f} \tag{A.35}$$

and consider the Lyapunov function

$$\bar{V} = \frac{\lambda_1}{2} \bar{q}^T P_q \bar{q} + \frac{1}{2} \bar{z}^T \bar{z} + \frac{\lambda_3}{2} \bar{\chi}^T P_0 \bar{\chi} + \frac{\lambda_2 \lambda_3}{2} \bar{\xi}_0^T P_0 \bar{\xi}_0 \tag{A.36}$$

where λ_1,λ_2,λ_3 are chosen as before. Using (A.32) and continuity of $u(t)$,

$$\frac{\dot{m}_f}{m_f} = \lim_{\tau\to 0}\frac{m_f(t+\tau)-m_f(t)}{\tau m_f(t)} \geq \lim_{\tau\to 0}\frac{e^{-\frac{\delta}{2}\tau}-1}{\tau} = -\frac{\delta}{2} \tag{A.37}$$

Using (A.14), (A.37) and the fact that

$$\dot{\bar{\nu}} \triangleq \frac{d}{dt}\frac{\nu}{m_f} = \frac{\dot{\nu}}{m_f} - \frac{\dot{m}_f}{m_f^2}\nu$$

for an arbitrary function ν, the derivative of $\bar{V}$ can be computed as

$$\begin{aligned}\dot{\bar{V}} \leq\; & -(\lambda_3\lambda_{20} - \frac{\delta}{2}\|P_0\|_2)|\bar{\xi}_0|^2 - (\lambda_{10} - \frac{\delta}{2}\,\|\|P_q\|_2\|_\infty)|\bar{q}|^2 - (\lambda_{30} - \frac{\delta}{2}\|P_0\|_2)|\bar{\chi}|^2 - (c_0 - \frac{\delta}{2})|\bar{z}|^2 \\ & + (\frac{\kappa_1^2}{d_1} + \sum_{i=2}^{\rho}\frac{\left(\frac{1}{\beta_0}+2\right)}{d_i})\frac{\eta_u^2}{m_f^2}\end{aligned} \tag{A.38}$$

Therefore, $\sqrt{\bar{V}} \in \mathcal{L}_\infty \cap \mathcal{S}(\Delta)$, all the normalized signals are bounded and small in the order of Δ in the mean square sense. In particular, $\bar{z} \in \mathcal{S}(\Delta)$.

On the other hand, following equations (3.18)-(3.24), (4.12)-(4.15), we can represent the control law (3.26), or (4.16) in the form

$$u = \sum_{i=1}^{\rho} L_i(\theta)z_i + \sum_{i=0}^{m} F_i(\theta)\frac{s^i}{K(s)}u + \sum_{i=0}^{n} G_i(\theta)\frac{s^i}{K(s)}y + \sum_{i=0}^{\rho} H_i(\theta)s^i y_r \tag{A.39}$$

for some bounded $\mathcal{C}^\infty$ functions L_i, F_i, G_i, H_i.

Substituting

$$u = Z_l(s,t)^{-1}R_l(s,t)y$$

in (A.39),we get

$$u = \sum_{i=1}^{\rho} L_i z_i + H_y(s,t)y + H_r(s,t)r \tag{A.40}$$

where $r = W_m(s)y_r$, $W_m(s-\delta/2)$ is a Hurwitz polynomial of order ρ, $H_y(s,t) = \sum_{i=0}^{m} F_i s^i (Z_l K)^{-1} R_l + \sum_{i=0}^{n} G_i s^i K^{-1}$, $H_r(s,t) = \sum_{i=0}^{\rho} H_i(\theta)\frac{s^i}{W_m(s)}$. Hence applying Theorem (22), pp.113 of [29], we obtain

$$\begin{aligned}\|(u)_t\|_{2\delta} \leq\; & \sum_{i=1}^{\rho} L_i\|(z_i)_t\|_{2\delta} + \left(\sup_{0\leq t_1\leq t}\int_0^{t_1}|h_{y,\delta/2}(t_1,\tau)|d\tau \sup_{0\leq t_1\leq t}\int_{t_1}^{t}|h_{y,\delta/2}(\tau,t_1)|d\tau\right)^{1/2}\|(y)_t\|_{2\delta} \\ & + \left(\sup_{0\leq t_1\leq t}\int_0^{t_1}|h_{r,\delta/2}(t_1,\tau)|d\tau \sup_{0\leq t_1\leq t}\int_{t_1}^{t}|h_{r,\delta/2}(\tau,t_1)|d\tau\right)^{1/2}\|(r)_t\|_{2\delta}\end{aligned}$$

where $h_{y,\delta/2}(t,\tau)$ and $h_{r,\delta/2}(t,\tau)$ are impulse responses of $H_y(s-\frac{\delta}{2},t)$ and $H_r(s-\frac{\delta}{2},t)$, respectively. Since $K(s-\delta/2)^{-1}$ and $W_m(s-\delta/2)^{-1}$ are both exponentially stable and L_i, F_i, G_i, H_i are bounded smooth functions of θ which is a bounded function of time, the supremum terms above are all finite. Hence, using $y = y_r + z_1$, we get

$$m_f^2 \leq c_m\left(1 + \sum_{i=1}^{\rho}\|(z_i)_t\|_{2\delta}^2\right) = c_m(1 + \|(z)_t\|_{2\delta}^2) = c_m(1 + \|(\bar{z}m_f)_t\|_{2\delta}^2)$$

or

$$m_f^2(t) \le c_m \left(1 + \int_0^t e^{-\delta(t-\tau)} |\bar{z}(\tau)|^2 m_f^2(\tau) d\tau\right) \tag{A.41}$$

where $c_m > 0$ is some constant. Applying the Bellman-Gronwall Lemma, we have

$$m_f^2(t) \le c_m + c_m^2 \int_0^t e^{-\delta(t-\tau_1)} |\bar{z}(\tau_1)|^2 e^{\int_{\tau_1}^t c_m |\bar{z}(\tau_2)|^2 d\tau_2} d\tau_1 \tag{A.42}$$

Now if

$$\int_t^{t+T} \Delta(\tau) d\tau \le \mu T + c_\mu, \ \forall t, T > 0$$

where $0 \le \mu < \mu^* \triangleq \frac{\delta}{c_m}$, then $\bar{z} \in \mathcal{S}(\mu)$ and $m_f \in \mathcal{L}_\infty$.

Once m_f is bounded, z is bounded. Then we apply the same argument as before, and conclude that all the closed loop signals are uniformly bounded, and the closed loop system is internally stable. In addition, all the error signals satisfy $z_i = \bar{z}_i m_f \in \mathcal{S}(\mu)$. That is, all the error signals, including the tracking error z_1 are of the order of μ in the mean square sense, i.e.

$$\int_t^{t+T} z_i^2(\tau) d\tau \le s_0 \mu T + s_1, \tag{A.43}$$

for any $t, T > 0$ and some constants $s_0, s_1 > 0$ independent of μ. ∎

Appendix B: Proof of Theorem 6.1

Following equations (6.15)-(6.19), it can be easily shown that the control law (6.20) can be represented as

$$u = \sum_{i=1}^{\rho} L_i(\hat{\theta}_u) z_i + \sum_{i=0}^{m} F_i(\hat{\theta}_u) \frac{s^i}{K(s)} u + \sum_{i=0}^{n} G_i(\hat{\theta}_u) \frac{s^i}{K(s)} y + \sum_{i=0}^{\rho} H_i(\hat{\theta}_u) s^i y_r \tag{B.1}$$

for some continuous functions L_i, F_i, G_i, H_i. On the other hand, from (2.4), we get

$$u = Z_l(s,t)^{-1} R_l(s,t) y \tag{B.2}$$

Substituting (B.2) into (B.1), we obtain

$$u = \sum_{i=1}^{\rho} L_i z_i + H_y(s,t) y + H_r(s,t) r \tag{B.3}$$

where $r = W_m(s) y_r$ so that $W_m(s - \delta/2)$ is some Hurwitz polynomial of order ρ, $H_r(s,t) = \sum_{i=0}^m H_i \frac{s^i}{W_m(s)}$, and $H_y(s,t) = \sum_{i=0}^m F_i s^i (Z_l(s,t)K(s))^{-1} R_l + \sum_{i=0}^n G_i s^i K(s)^{-1}$. Let us define a fictitious normalizing signal

$$m_f = \sqrt{1 + \|(u)_t\|_{2\delta}^2 + \|(y)_t\|_{2\delta}^2} \tag{B.4}$$

where $\delta \in (0,\ \delta_0)$ is a constant such that $Z_l(t, s - \frac{\delta}{2})^{-1}$ is an exponentially stable operator. Applying the arguments we have used in the proof of Theorem 5.1 again, we derive that

$$m_f^2 \le 1 + C\|r\|_\infty^2 + C\|(z)_t\|_{2\delta}^2 \tag{B.5}$$

where C is some finite positive constant. Since

$$|z| = \left|\frac{z}{m_s}\right| m_s \le \sqrt{1+\delta_1}\left|\frac{z}{m_s}\right| m_f \tag{B.6}$$

and

$$\frac{z}{m_s} = \epsilon - \frac{\phi}{m_s} \in \mathcal{L}_\infty \cap \mathcal{S}(\mu) \tag{B.7}$$

we have

$$m_f^2 \le 1 + C\|r\|_\infty^2 + C\|g^2 m_f^2\|_{2\delta}^2 \tag{B.8}$$

where $g = \sqrt{1+\delta_1}\frac{|z|}{m_s} \in \mathcal{S}(\mu)$, i.e., $\exists c_\mu, c_c > 0$ such that

$$\int_t^{t+T} g(\tau)^2 d\tau \le c_\mu \mu T + c_c \quad \forall t, T \ge 0$$

Applying the Bellman-Gronwall lemma, we obtain

$$m_f^2 \le (1 + C\|r\|_\infty^2)\left(1 + C\int_0^t e^{-(\delta - Cc_\mu\mu)(t-\tau)+c_c} g^2(\tau)d\tau\right) \tag{B.9}$$

Let $\mu^* = \frac{\delta}{Cc_\mu}$, then $\forall \mu \in (0,\ \mu^*)$, we have $m_f \in \mathcal{L}_\infty$. Since m_f bounds m_s which bounds all the closed loop signals, it follows that all signals are uniformly bounded. In addition, the tracking error satisfies $z_1 = \frac{z_1}{m_s} m_s \in \mathcal{S}(\mu)$.

In order to get some quantitative results, we first derive some performance bounds for the estimation error ϵ. We start by calculating the bound of the Lyapunov function V_θ. Noting that θ_c is assumed to be zero for simplicity and without loss of generality, (6.42) yields

$$V_\theta \le \frac{2\sigma_0 M^2 + \sup_{t\ge 0}\left(\frac{\tilde{x}_2^2}{d_0 m_s^2} + \frac{\|\Gamma^{-1}\|_2^2 |\dot{\theta}_u|^2}{\sigma_0}\right)}{\min\{2c_0, \frac{\sigma_0\|\Gamma\|_2}{2}\}} + V_\theta(0) \triangleq V_{max} \tag{B.10}$$

From (B.10), we notice that V_{max} is a function which decreases with increasing $c_0, d_0, \|\Gamma\|_2$. Therefore the bound on $|\epsilon|$, $|\tilde{\theta}_u|$ is decreasing with the increase of $c_0, d_0, \|\Gamma\|_2$. Using the fact that $\sqrt{|\dot{\theta}_u|}, \frac{|\tilde{x}_2|}{m_s} \in \mathcal{S}(\mu)\cap\mathcal{L}_\infty$ and $V_\theta(t) \ge 0, \forall t$, we integrate (6.41) and get

$$\int_t^{t+T} |\epsilon|^2 d\tau \le \frac{1}{c_0}\left(\left(\frac{1}{d_0} + \|\Gamma^{-1}\|_2 + 1\right) c\mu T + V_{max}\right), \quad \forall t, T \ge 0 \tag{B.11}$$

$$\int_t^{t+T} |b_z^T \epsilon|^2 d\tau \le \frac{4}{d_0}\left(\left(\frac{1}{d_0} + \|\Gamma^{-1}\|_2 + 1\right) c\mu T + V_{max}\right), \quad \forall t, T \ge 0 \tag{B.12}$$

where c is some positive constant independent of c_0, d_0, Γ.

Next, we derive some performance bounds for the auxiliary signal ϕ. Considering the quadratic function

$$V_\phi = \frac{1}{2}\left|\frac{\phi}{m_s}\right|^2 \tag{B.13}$$

it follows directly from (6.43) and the definition of s_i^2 that

$$\dot{V}_\phi \le -2c_0 V_\phi + \frac{\|\Xi\|^2 + 2\|\Upsilon\|^2 + |\zeta|^2 + |\varsigma|^2 + |y|^2}{4d_0 m_s^2}|\dot{\hat{\theta}}_u|^2 \tag{B.14}$$

Therefore

$$V_\phi \leq \frac{\sup_{t\geq 0}\left(\frac{\|\Xi\|^2+2\|\Upsilon\|^2+|\zeta|^2+|\varsigma|^2+|y|^2}{4d_0 m_s^2}|\dot{\hat{\theta}}_u|^2\right)}{2c_0} + V_\phi(0) \stackrel{\triangle}{=} V_{\phi,max} \tag{B.15}$$

Since Ξ,Υ,ζ,ς depend only on u, y, k, Ψ and $|\tilde{\theta}_u|$ is non-increasing with the increase of $c_0, d_0, \|\Gamma\|_2$, using (6.39) and (B.12), we obtain

$$\int_t^{t+T}\left|\frac{\phi}{m_s}\right|^2 d\tau \leq \frac{1}{c_0}\left(\left(1+\frac{1}{d_0}+\frac{1}{d_0^2}+\frac{\|\Gamma^{-1}\|_2}{d_0}\right)\frac{\bar{c}\|\Gamma\|_2^2}{d_0}\mu T + \frac{\bar{c}\|\Gamma\|_2^2}{d_0}V_{max} + V_{\phi,max}\right), \quad \forall t, T \geq 0 \tag{B.16}$$

where $\bar{c}$ is some positive constant independent of c_0, d_0, Γ. Finally, combining (B.11) and (B.16) we obtain (6.44). ■

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